

KINETICS OF ELECTRON-POSITRON PAIR PLASMAS USING AN ADAPTIVE MONTE CARLO METHOD

RAVI P. PILLA AND JACOB SHAHAM¹

Department of Physics, Columbia University, 538 West 120th Street, New York, NY 10027;
 ravi@cuphyb.phys.columbia.edu.

Received 1996 June 17; accepted 1997 April 4

ABSTRACT

A new algorithm for implementing the adaptive Monte Carlo method is given. It is used to solve the Boltzmann equations that describe the time evolution of a nonequilibrium electron-positron pair plasma containing high-energy photons. These are coupled nonlinear integro-differential equations. The collision kernels for the photons as well as pairs are evaluated for Compton scattering, pair annihilation and creation, bremsstrahlung, and Coulomb collisions. They are given as multidimensional integrals which are valid for all energies. For an homogeneous and isotropic plasma with no particle escape, the equilibrium solution is expressed analytically in terms of the initial conditions. For two specific cases, for which the photon and the pair spectra are initially constant or have a power-law distribution within the given limits, the time evolution of the plasma is analyzed using the new method. The final spectra are found to be in a good agreement with the analytical solutions. The new algorithm is faster than the Monte Carlo scheme based on uniform sampling and more flexible than the numerical methods used in the past, which do not involve Monte Carlo sampling. It is also found to be very stable. Some astrophysical applications of this technique are discussed.

Subject headings: elementary particles — plasmas — radiation mechanisms: nonthermal — relativity

1. INTRODUCTION

Nonthermal emission of high-energy radiation from a variety of compact astrophysical objects, e.g., γ -ray-burst sources (Mészáros & Rees 1993a, 1993b), pulsars (Chen & Ruderman 1993), active galactic nuclei (AGN; Lightman & Zdziarski 1987; Svensson 1994; and Padovani 1996), and jets in the AGN (Sikora 1994) seem to indicate the presence of a relativistic electron-positron pair plasma in the dense radiation fields of those sources. Such plasmas may exist also in the accretion disk coronas of the Galactic X-ray binaries (Sunyaev et al. 1992), the ergospheres of Kerr black holes (Piran & Shaham 1977), and the black hole accretion disks (Tanaka & Kusunose 1985; Björnsson & Svensson 1992). It is conceivable that the pair plasma in some of these sources is in thermodynamic equilibrium with itself and probably in equilibrium with the radiation. However, it is more plausible that many of them may consist of nonequilibrium pair plasmas (Coppi & Blandford 1990, hereafter CB90; Zdziarski 1988, 1989). Many of the previous papers on this topic have concentrated on the properties of a relativistic pair plasma in thermal equilibrium (e.g., Bisnovatyi-Kogan, Zeldovich, & Sunyaev 1971; Lightman & Band 1981; Lightman 1981, 1982; Svensson 1982b, hereafter S82b; Zdziarski 1985). Examples of the time evolution of a thermal pair plasma, taking into account the finite-medium radiative transfer effects can be found in Guilbert & Stepney (1985), Kusunose (1987), and Carrigan & Katz (1992). There are not many papers that deal with the evolution of a nonequilibrium pair plasma in detail; some examples can be found in Lightman & Zdziarski (1987), Svensson (1987), Zdziarski, Coppi, & Lamb (1990), and Coppi (1992, hereafter C92).

These investigations are generally based on the Monte Carlo (MC) methods or on solving the Boltzmann equations (kinetic theory approach). In a simple MC method based on uniform sampling (Pozdnyakov, Sobol, & Sunyaev 1977), individual particles are followed as they undergo interactions in the source. In this method, it is usually easy to take into account the spatial inhomogeneities and radiative transfer effects well. But it typically suffers from relatively poor photon statistics at higher energies and does not lend itself to time evolution calculations involving broadband spectra. For examples, of such MC simulations, see Novikov & Stern (1986). In the kinetic theory approach, the system is represented by the photon and particle distribution functions which are discretized in energy as well as the spatial coordinates, and the time evolution is determined by solving the Boltzmann equations numerically. In general, it is very difficult to solve the resulting integro-differential equations. Moreover, they are usually “stiff” (i.e., there are very different timescales in the problem). The principal advantage of this approach is that it gives good photon statistics at higher energies. Some examples of the kinetic theory approach can be found in C92, Ghisellini (1987), Svensson (1987), and Fabian et al. (1986).

There have been some attempts to improve the photon statistics in the conventional MC schemes which go by the name phase-space density (PSD) array representation. In this approach, the system is represented by the discretized distribution functions (as in the kinetic theory approach), but the particle or photon transitions between the energy bins is simulated using the MC method and the interaction between the spatial cells is modeled with the aid of the escape probabilities. So far this approach has been used to model only homogeneous and spherically symmetric systems (e.g., Stern 1985). Another recent variant of the MC

¹ Dr Shaham passed away during the course of this work.

method is based on the large-particle (LP) representation, which is described in detail by Stern et al. (1995). In this scheme, the system is represented by an array of “large particles,” each of which corresponds to a group of real particles sharing the same physical parameters (i.e., particle type, position, momentum, and energy). It is more flexible than the PSD approach in the sense that each LP is tagged with a statistical weight, which is proportional to the number of real particles represented by that LP. For example, this weight can be assigned based on the total energy carried by each LP. In many nonequilibrium systems of interest in astrophysics, the number of particles in the low-energy range is typically several orders of magnitude larger than that of the particles in the high-energy range. Therefore, the efficiency of the method may be improved by assigning lower statistical weight to the low-energy LPs. Intuitively this approach makes sense, but there is no general proof for its validity or effectiveness (except for the numerical experiments presented by Stern et al. 1995). Besides, the statistical weights are rather ad hoc.

From the preceding discussion, it is clear that the main problem in the analysis of nonequilibrium pair plasmas is the computational difficulty. The principal aim of this paper is to present a new method for solving the kinetic equations based on an adaptive MC sampling scheme. It is faster than the conventional MC method (based on uniform sampling) and is more flexible (and in some cases, faster) than the numerical methods previously used. Our method resembles the LP method described above, in the usage of the statistical weights, but it is much more rigorous. Moreover, it can accommodate anisotropic distributions with greater ease.

In a relativistic plasma containing arbitrary densities of pairs and the high-energy photons, the collision cross sections for various microscopic processes depend on the energy. One cannot use, for example, the simple Thomson cross section as one can do in the nonrelativistic case. In addition, there is a creation and annihilation of the pairs and photons that alter the densities. Therefore we have to follow the time evolution of the number density as well as the spectrum of each species. Besides, the problem is inherently nonlinear due to the form of the collision kernels in the Boltzmann equations. It is possible to write all the collision kernels as multidimensional integrals. This reduces the problem of solving the coupled Boltzmann equations for the photons and the pairs into a purely computational task of evaluating many of these integrals, after each time step, quickly and efficiently. This way of formulating the problem of kinetic theory is more flexible in accommodating any kind of distribution functions. We have developed a new algorithm, based on Monte Carlo sampling, for computing such integrals. The functional form of the integrands is not assumed a priori. Also, no constraint is placed on the shape of the integration region. Usually such integrals are evaluated either numerically (by using an equally spaced discrete grid) or through a Monte Carlo sampling technique. In order to make the former method more efficient, we have to choose the shape of the discrete mesh depending on the form of the integrand. This takes away the flexibility from the method (i.e., the algorithm will depend on the form of the integrand). The latter method, based on uniform sampling throughout the integration region, is widely used in astrophysics. It is possible to speed up the computation in this method by sampling selectively, i.e., sampling more fre-

quently in those domains where the integrand is larger. This scheme is known as the importance sampling method or the adaptive Monte Carlo method. There is an algorithm, originally due to Lepage (1978), which implements this. However, it is not well suited for the type of integrals that arise in the present context. We have developed a new algorithm to implement the adaptive Monte Carlo method which is very efficient (see below).

In the next section we define various quantities, explain the basic pair plasma model we use, and write down the general kinetic equations. In §§ 3 and 4 we give the integral expressions for various collision kernels, which are valid for all energies. These collision integrals are cast in a form that is well suited for the Monte Carlo integration. In § 5 we describe how we integrate the Boltzmann equations numerically. There we explain the adaptive Monte Carlo algorithm we use. In § 6 we express the final equilibrium state of an homogeneous and isotropic plasma (with no escape of particles or photons) analytically in terms of the initial conditions. Then we apply our time evolution code to two specific examples of nonequilibrium configurations and compare the final results with the corresponding analytical solutions. These examples serve as a test for the overall formalism presented in this paper. Finally, in § 7, we summarize this work and discuss some astrophysical applications.

2. MODEL, DEFINITIONS, AND THE NOTATION

We consider a neutral, stationary, and unmagnetized pair plasma which is nonthermal (i.e., not in equilibrium). We assume that the plasma is homogeneous and isotropic. If the plasma is in a moving source we must interpret all the physical quantities given below as the comoving-frame quantities. The number densities (i.e., the number of particles per unit volume) of the electrons, positrons, photons, and protons are given by n_- , n_+ , n_γ , and n_p , respectively ($n_- = n_+ + n_p$). Throughout this paper we express the momentum and energy in units of mc and mc^2 , respectively. Here m is the electron rest mass and c is the speed of light in free space. Therefore the momenta and the energies of the particles, as well as the photons, are represented by dimensionless numbers everywhere. For the models we consider here the protons can be assumed to be at rest. We assume that the state of the plasma is completely described by the Lorentz invariant distribution functions $f_\pm(x, p)$ and $f_\gamma(x, p)$, for positrons, electrons, and photons, respectively. Here x, p represent the position and the momentum four-vectors, respectively, and x, p represent the corresponding three-vectors. Our choice of the metric is such that $p^2 = 1$ for electrons. In the case of photons we have $p = \epsilon(1, \mathbf{k})$, where ϵ is the photon energy and \mathbf{k} is a unit vector in the direction of its three-momentum. Similarly, $p = \gamma(1, \boldsymbol{\beta})$ for the pairs. Here γ is the Lorentz factor and $\boldsymbol{\beta}$ is the velocity in units of c . We denote the magnitude of $\boldsymbol{\beta}$ by β . The number density of the particles of type i with a momentum p is given by $f_i d^3p$. We define the total densities of various species to be $n_i = \int f_i(p) d^3p$, where the integration extends over all values of the momenta. Because of the isotropy, we have $d^3p = 4\pi\epsilon^2 d\epsilon$ in the case of photons and $d^3p = 4\pi\beta\gamma^2 d\gamma$ for the pairs.

Since we assume that the plasma is homogeneous and isotropic, various distribution functions depend only on time and the energy (or the magnitude of the momentum). We define the spectral functions for photons, positrons, and

electrons to be

$$F_\gamma(\epsilon) = \frac{4\pi\epsilon^2}{n_\gamma} f_\gamma(\epsilon) \quad \text{and} \quad F_\pm(\gamma) = \frac{4\pi\beta\gamma^2}{n_\pm} f_\pm(\gamma), \quad (2.1)$$

respectively. The time dependence of these functions is not shown explicitly. The spectral functions are normalized so that

$$\int_0^\infty d\epsilon F_\gamma(\epsilon) = 1 \quad \text{and} \quad \int_1^\infty d\gamma F_\pm(\gamma) = 1. \quad (2.2)$$

We see that the number of photons of energy ϵ per unit volume and unit energy is given by $n_\gamma F_\gamma(\epsilon)$. We will assume that the electrons and the positrons have the same spectral functions, i.e., $F_-(\gamma) = F_+(\gamma)$ for all values of γ , which we denote by $F_e(\gamma)$.

The equilibrium spectral functions, which are independent of time, are given by

$$F_\gamma(\epsilon) = \frac{1}{2\zeta(3)\Theta^3} \frac{\epsilon^2}{\exp(\epsilon/\Theta) - 1}, \quad (2.3)$$

$$F_e(\gamma) = \frac{1}{\Theta K_2(1/\Theta)} \beta\gamma^2 \exp(-\gamma/\Theta). \quad (2.4)$$

Equation (2.3) comes from the Planck function for the photons, where ζ is the Riemann zeta function and $\zeta(3) \cong 1.202$. In that equation we have used the equilibrium density of photons

$$n_\gamma = 16\pi\zeta(3) \left(\frac{mc}{h} \Theta \right)^3, \quad (2.5)$$

where h is the Planck's constant. Equation (2.4) is the relativistic Maxwell-Boltzmann distribution for electrons and K_2 is the second-order modified Bessel function of the second kind. In all these equations $\Theta = k_B T/mc^2$ is the dimensionless temperature of the plasma, where T is the temperature and k_B is the Boltzmann constant.

To study the time evolution of this system we should proceed from the relativistic Boltzmann equations for the pairs and photons. In the latter case it is the same as the radiative transfer equation. The Boltzmann equation (see e.g., de Groot, van Leeuwen, & van Weert 1980) for the particles of type i , described by f_i , which takes into account the collisions with the particles of type j , described by f_j , is given by

$$p^\mu \partial_\mu f_i(x, p) = \sum_j \int \frac{d^3q}{q^0} d\Omega' [f_i(x, p') f_j(x, q') - f_i(x, p) f_j(x, q)] F \sigma_{ij}. \quad (2.6)$$

Here ∂_μ is the partial derivative with respect to x^μ and the summation over μ is implied. The summation for j extends over all relevant processes. Here q^0 is the energy component of the four-vector q . Using the initial and the final momenta to designate the particles, the collision processes can be represented as $p + q \leftrightarrow p' + q'$. The solid angle around one of the outgoing particles is $d\Omega'$. Finally, σ_{ij} is the cross section for the process and F is the invariant flux factor. It is necessary to remark that in the present form, the above equation cannot account for the quantum mechanical Bose enhancement and Fermi blocking effects, respectively, for the photons and pairs. In order to do so, we need to take into account the particle occupation numbers in the phase

space. For photons, this is given by

$$g_\gamma(\epsilon) = \frac{1}{2} \left(\frac{h}{mc} \right)^3 f_\gamma(\epsilon) = \left(\frac{h}{mc} \right)^3 \frac{n_\gamma F_\gamma(\epsilon)}{8\pi\epsilon^2}, \quad (2.7)$$

which in the equilibrium case reduces to $1/[\exp(\epsilon/\Theta) - 1]$, as expected. If we are considering a process in which two particles of momenta p and q produce a photon of momentum p' , then we should make the replacement $f_i(p)f_j(q) \rightarrow f_i(p)f_j(q)[1 + g_\gamma(p')]$ in the Boltzmann equation. These effects play a significant role only when $g_\gamma \simeq 1$ or $n_\gamma \epsilon^{-2} F_\gamma(\epsilon) \simeq 1.76 \times 10^{30} \text{ cm}^{-3}$. For the densities and the energies of interest here, these quantum mechanical effects can be neglected. An analogous remark applies to the case of the pairs. Such induced effects in a relativistic thermal plasma at high temperatures and densities have been considered by many authors in the past (e.g., Ramaty, McKinley, & Jones 1982).

The Boltzmann equations reduce to simple rate equations in the comoving frame as a result of the homogeneity and isotropy of the plasma. We denote the comoving time coordinate by t . The rate equations are given by

$$\frac{\partial}{\partial t} f_i = \sum_q [\eta_i - f_i \chi_i]_q, \quad (2.8)$$

where i stands for either photons or electrons and q labels the binary collision process (Compton scattering, pair processes, bremsstrahlung, or Coulomb collisions). The summation runs over all those processes that involve a particle of type i among the products of the collision. Here η_i is the emission coefficient for the production of a particle of type i with momentum p (or scattering of such a particle into that final momentum state) and χ_i is the corresponding absorption coefficient. Notice that f_i , η_i , and χ_i depend only on the energy of the particles and time. In order to obtain the collision kernels, η_i and χ_i , we require the binary reaction rates in a relativistic plasma (e.g., de Groot et al. 1980; Baring 1987a). Using the appropriate reaction rates we can write

$$\eta_i(p) = \sum_{l,m} \frac{c}{1 + \delta_{lm}} \int dF_{lm} \mathcal{F}_{lm} \frac{d\sigma_{lm}}{dP}, \quad (2.9)$$

where $\delta_{lm} = 1$ for identical colliding particles (i.e., $l = m$) and is zero otherwise. The summation in this equation is over those incident states (labeled by l and m) which result in a final state labeled by i . Furthermore, $d\sigma_{lm}/dP$ is the differential cross section for the process, whereas dP is a shorthand for d^3p which is defined above. The four-momenta of the colliding particles are given by $p_k = (p_k^0, \mathbf{p}_k)$ (for $k = l, m$) and the four-momentum of one of the outgoing particles is p . The product of the phase-space densities of the colliding particles dF_{lm} is given by

$$dF_{lm} = \prod_{j=l,m} f_j(p_j) d^3p_j. \quad (2.10)$$

We have $d^3p_l = \epsilon_l^2 d\epsilon_l d\Omega_l$ for the photons and $d^3p_l = \beta_l \gamma_l^2 d\Omega_l d\gamma_l$ for the pairs. The kinematic factor \mathcal{F}_{lm} for binary collisions (see, e.g., Landau & Lifshitz 1975) is given by

$$\mathcal{F}_{lm} = (u_l u_m) \beta_{\text{rel}}(p_l, p_m), \quad (2.11)$$

where $u_l = p_l/p_l^0$ and β_{rel} is the relative velocity of the colliding particles in units of c . If at least one of the colliding

particles is a photon we will have $\beta_{\text{rel}} = 1$. Otherwise

$$\beta_{\text{rel}}(p_l, p_m) = \frac{[(\beta_l - \beta_m)^2 - (\beta_l \times \beta_m)^2]^{1/2}}{1 - \beta_l \cdot \beta_m}. \quad (2.12)$$

The integration in equation (2.9) is over a region U of the phase space of the colliding particles, which is specified by the energy-momentum conservation. It depends on the energy p^0 of the final state. Now we specialize to the case of a process for which the reacting particles are labeled by $l = 1$ and $m = 2$. By using equation (2.1) we can express dF_{12} in terms of the spectral functions and the densities. This gives the following final expression for the emission coefficient (i.e., the production rate) for electrons or photons:

$$\eta(\epsilon) = \frac{cn_1 n_2}{16\pi^2(1 + \delta_{12})} \int_U \prod_{j=1}^2 [F_j(\epsilon_j) d\epsilon_j d\Omega_j] \mathcal{F}_{12} \frac{d\sigma}{dP}. \quad (2.13)$$

Now we define the total reaction rate between two particles of energies ϵ_1 and ϵ_2 to be

$$R(\epsilon_1, \epsilon_2) = \frac{cn_1 n_2}{2(1 + \delta_{12})} \int_{-1}^1 d\mu \mathcal{F}_{12} \sigma_{\text{total}}, \quad (2.14)$$

where μ is the cosine of the angle between the momenta of the colliding particles and σ_{total} is the total cross section for the process considered (integrated over the entire phase space of the emitted particle). Clearly \mathcal{F}_{12} as well σ_{total} depend only on ϵ_1 , ϵ_2 , and μ . Now it is possible to express the emission coefficient in terms of the total reaction rate as

$$\eta(\epsilon) = \int \prod_{j=1}^2 [d\epsilon_j F_j(\epsilon_j)] R(\epsilon_1, \epsilon_2) P(\epsilon_1, \epsilon_2; \epsilon), \quad (2.15)$$

where the integration is over all values of ϵ_1 and ϵ_2 without any restriction (in contrast to eq. [2.13]). In the above equation, P is the probability, integrated over all incident and emergent angles of the particles, for emitting a particle of energy ϵ , from a collision between the particles of energies ϵ_1 and ϵ_2 . It is normalized so that $\int d\epsilon P(\epsilon_1, \epsilon_2; \epsilon) = 1$, where the integration is over all values of ϵ . Equation (2.15) has been used by several previous authors (e.g., CB90).

We can obtain the absorption coefficient from equation (2.9) with only minor changes. For the absorption of the particles of type i with a momentum p_i we find that

$$f_i(p_i) \chi_i(p_i) = \sum_j \frac{c}{1 + \delta_{ij}} \int_U \frac{dF_{ij}}{dP_i} \mathcal{F}_{ij} \sigma_{\text{total}}. \quad (2.16)$$

Here σ_{total} is the total scattering cross section for the process. The summation extends over all relevant processes. For a binary process, involving the particles of type i and type j , the absorption coefficient can be written in terms of the spectral functions as follows:

$$\chi_i(\epsilon_i) = \frac{cn_j}{4\pi(1 + \delta_{ij})} \int_U d\epsilon_j d\Omega_j F_j(\epsilon_j) \mathcal{F}_{ij} \sigma_{\text{total}}, \quad (2.17)$$

where the integration region U is determined by the energy-momentum conservation. This way of writing the emission and absorption coefficients is very convenient for Monte Carlo evaluation we describe below.

We remark that in equations (2.13)–(2.17) we have used ϵ in a generic way and it has to be replaced by γ whenever it refers to the pairs. Physically, $4\pi\epsilon^2\eta(\epsilon)$ is the rate at which

photons of energy ϵ are emitted per unit volume and unit energy due to the process under consideration; similarly, $4\pi\beta\gamma^2\eta(\gamma)$ gives the corresponding electron emission rate (recall that we express energy in units of mc^2). Electron and photon absorption rates are obtained in a similar way. If the size of the system is l , the optical depth τ and the absorption coefficient are related by $\tau = l\chi/c$. Equations (2.13) and (2.17) constitute the point of departure for the following two sections where we obtain the emission and the absorption coefficients for the photon and the pair kinetic equations. We remark here that in the case of Compton scattering of the photons as well as the pairs, the collision integrals only give the rate at which the spectrum changes at a given energy and do not imply any change in the total numbers of the particles.

3. COLLISION INTEGRALS FOR PHOTONS

The preceding discussion has been very general. We now obtain the integral expressions for the photon emission coefficients due to Compton scattering, two-photon pair annihilation, and bremsstrahlung and the absorption coefficients due to Compton scattering and the pair creation. In this paper we do not consider the double-Compton emission or the three-photon emission through pair annihilation. Also we do not consider the effect of photon absorption through the inverse-bremsstrahlung (free-free absorption).

3.1. Compton Scattering of Photons

The problem of Comptonization in astrophysics has been analyzed extensively by many previous authors (e.g., Blumenthal & Gould 1970; Rybicki & Lightman 1979; and more recently by CB90). Here we obtain an integral expression which is valid at all energies of the incident electrons and photons. Throughout this paper we call the comoving frame of the plasma the C frame. Let p and p_1 be the momenta of the incident electron and photon, respectively, in the C frame. Let q and q_1 be the corresponding momenta after the scattering. Recall that $p^2 = 1$ and $p_1^2 = 0$. We require the final photon energy to be ϵ . Hence we set $q_1 = \epsilon(1, \mathbf{k})$, where \mathbf{k} is the directional unit vector. We write $p = \gamma(1, \boldsymbol{\beta})$ and $p_1 = \epsilon_1(1, \mathbf{k}_1)$. Here γ is the Lorentz factor of the incident electron, $\boldsymbol{\beta}$ is its three-velocity in units of c , ϵ_1 is the energy of the incident photon, and \mathbf{k}_1 is its directional unit vector. Using the fact that $(p + p_1 - q_1)^2 = q^2 = 1$ we obtain the well-known relation between the initial and the final photon energies, viz., $\epsilon = \tilde{\epsilon}(\epsilon_1)$ or $\epsilon_1 = \tilde{\epsilon}_1(\epsilon)$, where

$$\tilde{\epsilon} = \frac{a_1 \gamma \epsilon_1}{a\gamma + b\epsilon_1} \quad \text{and} \quad \tilde{\epsilon}_1 = \frac{a\gamma \epsilon}{a_1 \gamma - b\epsilon}. \quad (3.1)$$

Here $a = 1 - \boldsymbol{\beta} \cdot \mathbf{k}$, $a_1 = 1 - \boldsymbol{\beta} \cdot \mathbf{k}_1$, and $b = 1 - \cos \theta$, while $\cos \theta = \mathbf{k} \cdot \mathbf{k}_1$ gives the cosine of the photon scattering angle in the C frame. Let $\mu = \cos \theta$ and the cosine of the angle between $\boldsymbol{\beta}$ and \mathbf{k} is defined to be μ' . The angle between the planes formed by the pairs of vectors $(\mathbf{k}, \mathbf{k}_1)$ and $(\mathbf{k}, \boldsymbol{\beta})$ is defined to be ϕ . It is easy to show (see the Appendix for further details) using equation (2.13) that the Compton emissivity for photons is given by

$$\eta(\epsilon) = \frac{cn_-(n_- + n_+)r_e^2}{8\pi\epsilon^2} \int_U (d\gamma d\mu d\mu' d\phi) F_e(\gamma) F_\gamma(\tilde{\epsilon}_1) \left(\frac{\Delta}{2\gamma^2 a \zeta} \right), \quad (3.2)$$

where $\Delta = \xi^2 - \xi \sin^2 \theta' + 1$ and $\xi = a_1 \gamma / (a_1 \gamma - b\epsilon)$, while θ' is the photon scattering angle in the rest frame of the incident electron and r_e is the classical radius of an electron. The region of integration U is defined by $\gamma_{\min} \leq \gamma \leq \gamma_{\max}$, $-1 \leq \mu, \mu' \leq 1$, and $0 \leq \phi \leq 2\pi$ subject to the condition that $\epsilon_{1\min} \leq \epsilon_1 \leq \epsilon_{1\max}$. Here γ_{\min} and γ_{\max} are the limiting electron or positron energies in the plasma. Similarly $\epsilon_{1\min}$ and $\epsilon_{1\max}$ are the limiting photon energies.

Now we obtain the corresponding "absorption" coefficient (as stated before, this is not a real absorption; the photons are scattered into a different energy bin). Let $p = \epsilon(1, \mathbf{k})$ and $q = \gamma(1, \boldsymbol{\beta})$ be the initial momenta of the photon and the electron, respectively. Various symbols have the same meaning as above. The photon energy in the rest frame of the incident electron is given by $x = pq = \gamma\epsilon(1 - \beta\mu)$, where μ is the cosine of the angle between the vectors $\boldsymbol{\beta}$ and \mathbf{k} . Now $n_j = n_+ + n_-$, $\delta_{ij} = 0$, $F_j = F_e$, $d\Omega_j = 2\pi d\mu$, $\mathcal{F}_{ij} = (1 - \beta\mu)$, $\epsilon_i = \epsilon$, and $\epsilon_j = \gamma$. Substituting these expressions into equation (2.17), we obtain

$$\chi(\epsilon) = \frac{c(n_+ + n_-)}{2} \int_U d\mu d\gamma F_e(\gamma)(1 - \beta\mu)\sigma_{\text{total}}(x), \quad (3.3)$$

where

$$\sigma_{\text{total}}(x) = 2\pi r_e^2 \left\{ \frac{1+x}{x^3} \left[\frac{2x(1+x)}{1+2x} - \ln(1+2x) \right] + \frac{\ln(1+2x)}{2x} - \frac{1+3x}{(1+2x)^2} \right\} \quad (3.4)$$

is the total cross section for Compton scattering (e.g., Jauch & Rohrlich 1980, hereafter JR80). The integration domain U is defined by $\gamma_{\min} \leq \gamma \leq \gamma_{\max}$ and $-1 \leq \mu \leq 1$ without any restriction. Here γ_{\min} and γ_{\max} are the limiting electron energies, as in the previous case.

3.2. Effect of Pair Production/Annihilation on Photons

The emissivity due to the annihilation of relativistic electron-positron pairs (creating two photons) has been analyzed by many authors before (e.g., Zdziarski 1980; Ramaty & Mészáros 1981; Yahel & Brinkmann 1981; Svensson 1982a, hereafter S82a). We give here the final result using the notation of S82a and refer the reader to that paper for a detailed derivation. Let $p_i = \gamma_i(1, \boldsymbol{\beta}_i)$; $i = 1, 2$ be the momenta of the electron and the positron, respectively, in the C frame. Let $q_1 = \epsilon(1, \mathbf{k})$ be the momentum of one of the emitted photons. Here $c\boldsymbol{\beta}_i$ are the particle velocities and γ_i are the corresponding Lorentz factors, ϵ is the photon energy, and \mathbf{k} is its directional unit vector. The momentum of the C frame itself is denoted by $q = (1, \mathbf{0})$. We call the center-of-momentum frame of the pair the CM frame and the quantities in this frame appear with a suffix "cm." The particle momenta in this frame are $p_{1\text{cm}} = \gamma_{\text{cm}}(1, \boldsymbol{\beta}_{\text{cm}})$, $p_{2\text{cm}} = \gamma_{\text{cm}}(1, -\boldsymbol{\beta}_{\text{cm}})$, $q_{1\text{cm}} = \epsilon_{\text{cm}}(1, \mathbf{k}_{\text{cm}})$, and $q_{2\text{cm}} = \gamma_c(1, -\boldsymbol{\beta}_c)$. Here γ_{cm} is the Lorentz factor of the electron or positron, ϵ_{cm} is the photon energy, and \mathbf{k}_{cm} is its directional unit vector (in the CM frame). The velocity of the CM frame as measured in the C frame is $c\boldsymbol{\beta}_c$ and γ_c is the corresponding Lorentz factor. Various directional cosines are defined as follows: μ , x , y , and z are the cosines of the angles between the pairs of vectors $(\boldsymbol{\beta}_1, \boldsymbol{\beta}_2)$, $(\mathbf{k}_{\text{cm}}, \boldsymbol{\beta}_{\text{cm}})$, $(\boldsymbol{\beta}_c, \boldsymbol{\beta}_{\text{cm}})$, and $(\boldsymbol{\beta}_c, \mathbf{k}_{\text{cm}})$, respectively; the angle between the planes formed by the pairs of vectors $(\boldsymbol{\beta}_c, \mathbf{k}_{\text{cm}})$ and $(\boldsymbol{\beta}_c, \boldsymbol{\beta}_{\text{cm}})$ is denoted by ϕ . After analyzing the kinematics, we obtain

$\gamma_{\text{cm}} = \sqrt{[\frac{1}{2} + \gamma_1\gamma_2(1 - \beta_1\beta_2\mu)/2]}$, $\gamma_c = (\gamma_1 + \gamma_2)/(2\gamma_{\text{cm}})$, $y = (\gamma_1 - \gamma_2)/(2\beta_c\beta_{\text{cm}}\gamma_c\gamma_{\text{cm}})$, $z = (\epsilon - \gamma_c\gamma_{\text{cm}})/(\beta_c\gamma_c\gamma_{\text{cm}})$, and $x = yz + \sqrt{[(1-y^2)(1-z^2)]} \cos \phi$. Now using equation (2.13) we obtain the pair emissivity

$$\eta(\epsilon) = \frac{cn_+n_-}{4\pi\epsilon^2} \int_U d\mu d\phi \prod_{i=1}^2 [F_e(\gamma_i) d\gamma_i] \frac{\beta_{\text{cm}}\gamma_{\text{cm}}}{\beta_c\gamma_c\gamma_1\gamma_2} \left(\frac{d\phi}{d\Omega} \right)_{\text{cm}}. \quad (3.5)$$

The differential cross section in the CM frame is given by

$$\left(\frac{d\phi}{d\Omega} \right)_{\text{cm}} = \frac{r_e^2}{4\beta_{\text{cm}}\gamma_{\text{cm}}^2} \left[-1 + \frac{3 - \beta_{\text{cm}}^4}{2} (\zeta_+ + \zeta_-) - \frac{1}{2\gamma_{\text{cm}}^4} (\zeta_+^2 + \zeta_-^2) \right], \quad (3.6)$$

where $\zeta_{\pm} = 1/(1 \pm \beta_{\text{cm}}x)$. The integration domain U in equation (3.5) is given by $\gamma_{\min} \leq \gamma_{1,2} \leq \gamma_{\max}$, $-1 \leq \mu \leq 1$, and $0 \leq \phi \leq 2\pi$, subject to the condition $-1 \leq z \leq 1$, which is equivalent to the condition $\Gamma_-(\gamma_1, \gamma_2, \mu; \epsilon) \leq \gamma_{\text{cm}}(\gamma_1, \gamma_2, \mu) \leq \Gamma_+(\gamma_1, \gamma_2, \mu; \epsilon)$, where $\Gamma_{\pm} = \epsilon\gamma_c(1 \pm \beta_c)$. Here γ_{\min} and γ_{\max} are the limiting pair energies in the plasma.

Now we obtain the photon absorption coefficient due to pair creation. Let the initial momenta of the photons be $p = \epsilon(1, \mathbf{k})$ and $p' = \epsilon'(1, \mathbf{k}')$, with the usual meaning for various symbols. If an electron-positron pair is produced, then the CM-frame Lorentz factor of the electron is given by $\gamma_{\text{cm}} = \sqrt{[\epsilon\epsilon'(1 - \mu)/2]}$, where μ is the cosine of the angle between the vectors \mathbf{k} and \mathbf{k}' . Using equation (2.17) we find

$$\chi(\epsilon) = \frac{cn_{\gamma}}{4} \int_U d\mu d\epsilon' F_{\gamma}(\epsilon')(1 - \mu)\sigma_{\text{total}}(\gamma_{\text{cm}}). \quad (3.7)$$

Since $\sigma(\gamma\gamma \rightarrow ee) = 2\beta_{\text{cm}}^2\sigma(ee \rightarrow \gamma\gamma)$, by integrating equation (3.6), we find

$$\sigma_{\text{total}}(\gamma_{\text{cm}}) = \frac{\pi r_e^2 \beta_{\text{cm}}}{\gamma_{\text{cm}}^2} \left[\frac{(3 - \beta_{\text{cm}}^4)}{\beta_{\text{cm}}} \ln \left(\frac{1 + \beta_{\text{cm}}}{1 - \beta_{\text{cm}}} \right) - 2 - \frac{2}{\gamma_{\text{cm}}^2} \right]. \quad (3.8)$$

The integration domain U in equation (3.7) is defined by $-1 \leq \mu \leq 1$ and $\epsilon^* \leq \epsilon' \leq \epsilon_{\max}$, where $\epsilon^* = 2/[\epsilon(1 - \mu)]$ is the pair creation threshold energy and ϵ_{\max} is the limiting photon energy in the plasma.

3.3. Bremsstrahlung Emissivity

The bremsstrahlung emissivity of a pair plasma has been analyzed in several papers (e.g., Haug 1975b, 1985c, 1987, 1989; Dermer 1986). The final expression for the photon emissivity can be written as

$$\eta_{\text{pair}}(\epsilon) = \frac{c\alpha r_e^2}{8\pi^2\epsilon} \int_U d\mu d\Omega \prod_{i=1}^2 [F_e(\gamma_i) d\gamma_i] \frac{\mathcal{F}_{12}}{\rho} \times \left[\frac{1}{2} (n_+^2 + n_-^2) \frac{C_1}{\Delta_1} + n_+n_- \frac{C_2}{\Delta_2} \right], \quad (3.9)$$

where α is the fine-structure constant. The first term inside the brackets represents the sum of the electron-electron and the positron-positron contributions, and the second term gives the electron-positron contribution. The expressions for ρ , \mathcal{F}_{12} , C_i , and Δ_i , along with the definitions of the integration variables μ and Ω are given in the Appendix (see also Fig. 1). The emissivity due to pair-proton bremsstrah-

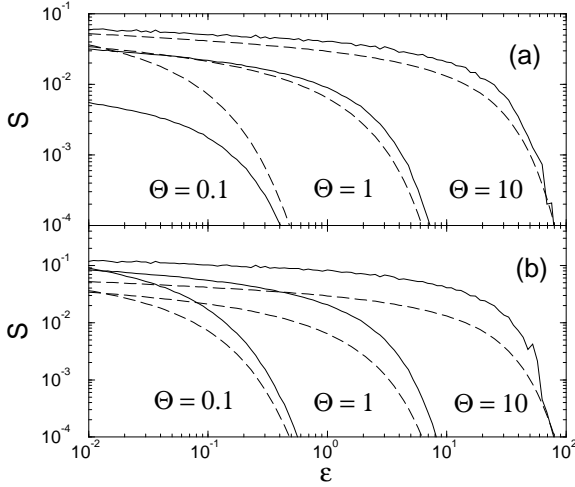


FIG. 1.—Emissivity due to (a) electron-electron and (b) electron-positron bremsstrahlung from a thermal plasma for three different temperatures. The dashed lines represent the emissivity due to pair-proton bremsstrahlung (given here for comparison). The energy of the emitted photon is ϵ and $S = 4\pi\epsilon^3\eta(\epsilon)/(cn_1n_2\sigma_{\text{Th}})$, where $n_{1,2}$ are the appropriate densities. These results agree with Haug (1985c) and Dermer (1986).

lung can be written as

$$\eta_{\text{proton}}(\epsilon) = \frac{cn_p(n_+ + n_-)}{4\pi\epsilon^2} \int_{1+\epsilon}^{\gamma_{\text{max}}} d\gamma F_e(\gamma) \beta \left(\frac{d\sigma}{d\epsilon} \right)_{\text{proton}}, \quad (3.10)$$

where $(d\sigma/d\epsilon)_{\text{proton}}$ is the cross section for this process (see e.g., JR80). Here the protons are assumed to be at rest.

4. COLLISION INTEGRALS FOR PAIRS

4.1. Compton Scattering of the Pairs

The effect of Compton scattering on the pair distribution can be analyzed in a manner similar to that of the photon Comptonization discussed in the previous section. Let p and p_1 be the momenta of the incident electron and the photon, respectively. Let q and q_1 be their corresponding momenta after the scattering. We require the final electron energy to be γ . Hence, we set $q = \gamma(1, \beta)$, where $c\beta$ is the velocity of the scattered electron. We write $p = \gamma_1(1, \beta_1)$, $p_1 = \epsilon_1(1, k_1)$, and $q_1 = \epsilon(1, k)$. Using $q_1^2 = (p + p_1 - q)^2 = 0$, we obtain a relation between the initial energy of the photon and the final energy of the electron given by $\epsilon_1 = \tilde{\epsilon}_1$, where

$$\tilde{\epsilon}_1 = \frac{b\gamma\gamma_1 - 1}{a_1\gamma_1 - a\gamma}. \quad (4.1)$$

In this equation $a = 1 - \beta \cdot k_1$, $a_1 = 1 - \beta_1 \cdot k_1$, and $b = 1 - \beta \cdot \beta_1$. Let μ be the cosine of the angle between the vectors β and k_1 . The cosine of the angle between the vectors β and β_1 is defined to be μ' . The angle between the planes formed by the pairs of vectors (β, k_1) and (β, β_1) is defined to be ϕ . Now the emission coefficient due to Compton scattering can be written (see the Appendix for details) as

$$\eta(\gamma) = cn_\gamma(n_- + n_+)r_e^2 \int du d\mu' d\phi d\gamma_1 F_e(\gamma_1) F_\gamma(\tilde{\epsilon}_1) \times \frac{a_1 X}{16\pi\epsilon\gamma\rho_1} \left| \frac{d\tilde{\epsilon}_1/d\gamma}{1 + d\epsilon/d\gamma} \right|, \quad (4.2)$$

where

$$X = \frac{\rho_1}{\rho_2} + \frac{\rho_2}{\rho_1} + 2\left(\frac{1}{\rho_1} - \frac{1}{\rho_2}\right) + \left(\frac{1}{\rho_1} - \frac{1}{\rho_2}\right)^2, \quad (4.3)$$

while $\rho_1 = a_1\tilde{\epsilon}_1\gamma_1$ and $\rho_2 = a\tilde{\epsilon}_1\gamma$. The integration region is given by $-1 \leq \mu, \mu' \leq 1$, $0 \leq \phi \leq 2\pi$, and $\gamma_{\text{min}} \leq \gamma_1 \leq \gamma_{\text{max}}$ subject to the condition that $\epsilon_{\text{min}} \leq \epsilon_1 \leq \epsilon_{\text{max}}$.

Next we consider the absorption coefficient due to Compton scattering. Let $p = \epsilon(1, k)$ and $q = \gamma(1, \beta)$ be the initial momenta of the photon and the electron, respectively. The photon energy in the rest frame of the incident electron is given by $x = pq = \gamma\epsilon(1 - \beta\mu)$, where μ is the cosine of the angle between the vectors β and k . As in § 3.1, it can be shown that

$$\chi(\gamma) = \frac{cn_\gamma}{2} \int d\mu d\epsilon F_\gamma(\epsilon)(1 - \beta\mu)\sigma_{\text{total}}(x), \quad (4.4)$$

where σ_{total} is given by equation (3.4). The integration domain is given by $-1 \leq \mu \leq 1$ and $\epsilon_{\text{min}} \leq \epsilon \leq \epsilon_{\text{max}}$.

4.2. Production and Annihilation of the Pairs

The analysis for this case is analogous to that for the pair annihilation emissivity discussed above. Let $p_i = \epsilon_i(1, k_i)$ be the momenta of the photons in the C frame, where ϵ_i are their energies and k_i are their directional unit vectors. Let $p = \gamma(1, \beta)$ be the momentum of one of the emitted particles. Here $c\beta$ is its velocity in the C frame and γ is the corresponding Lorentz factor. The momentum of the C frame itself is denoted by $q = (1, 0)$. We denote the CM-frame quantities with a suffix “cm.” Let $p_{1\text{cm}} = \epsilon_{\text{cm}}(1, k_{\text{cm}})$, $p_{2\text{cm}} = \epsilon_{\text{cm}}(1, -k_{\text{cm}})$, $p_{\text{cm}} = \gamma_{\text{cm}}(1, \beta_{\text{cm}})$, and $q_{\text{cm}} = \gamma_c(1, \beta_c)$ represent p_1 , p_2 , p , and q , respectively, in the CM frame. The velocity of the C frame as measured in the CM frame is $c\beta_c$ and γ_c is the corresponding Lorentz factor. Various directional cosines are defined as follows: μ , x , y , and z are the cosines of the angles between the pairs of vectors (k_1, k_2) , $(k_{\text{cm}}, \beta_{\text{cm}})$, (k_{cm}, β_c) , and $(\beta_{\text{cm}}, \beta_c)$, respectively. The angle between the planes formed by the pairs of vectors $(\beta_c, \beta_{\text{cm}})$ and (β_c, k_{cm}) is defined to be ϕ . We have $\gamma_{\text{cm}} = \epsilon_{\text{cm}} = [\epsilon_1\epsilon_2(1 - \mu)/2]^{1/2}$, $\gamma_c = (\epsilon_1 + \epsilon_2)/(2\epsilon_{\text{cm}})$, $y = (\epsilon_2 - \epsilon_1)/(2\beta_c\gamma_c\epsilon_{\text{cm}})$, $z = (\gamma_c\gamma_{\text{cm}} - \gamma)/\Delta$, whereas $\Delta = \beta_c\beta_{\text{cm}}\gamma_c\gamma_{\text{cm}}$, and $x = yz + [(1 - y^2)(1 - z^2)]^{1/2} \cos \phi$. We can now write (see the Appendix for more details) the pair creation emissivity as

$$\eta(\gamma) = \frac{cn_\gamma^2}{16\pi\beta\gamma^2} \int_U d\mu d\phi \prod_{i=1}^2 [F_\gamma(\epsilon_i)d\epsilon_i] \frac{1 - \mu}{\Delta} \left(\frac{d\sigma}{d\Omega} \right)_{\text{cm}}, \quad (4.5)$$

where the differential cross section is obtained by multiplying the one given by equation (3.6) with β_{cm}^2 . The integration domain is given by $\epsilon_{\text{min}} \leq \epsilon_{1,2} \leq \epsilon_{\text{max}}$, $-1 \leq \mu \leq 1$, and $0 \leq \phi \leq 2\pi$, subject to the condition $-1 \leq z \leq 1$, which is equivalent to $\Gamma_- \leq \gamma \leq \Gamma_+$, where $\Gamma_{\pm} = \gamma_c\gamma_{\text{cm}}(1 \pm \beta_c\beta_{\text{cm}})$.

For the absorption coefficient due to pair creation, consider an electron of momentum $p = \gamma(1, \beta)$ annihilating with a positron of momentum $p' = \gamma'(1, \beta')$. Their common Lorentz factor in the CM frame is given by $\gamma_{\text{cm}} = [\gamma\gamma'(1 - \beta\beta'\mu)/2]^{1/2}$, where μ is the cosine of the angle between the vectors β and β' . Setting $\epsilon_i = \gamma$, $\epsilon_j = \gamma'$, $n_j = n_{\pm}$, $\delta_{ij} = 0$, $d\Omega_j = 2\pi d\mu$, $F_j = F_e$, and $\mathcal{F}_{ij} = \beta_r\gamma_r(\gamma\gamma')^{-1}$ in

equation (2.17) we find

$$\chi_{\pm}(\gamma) = \frac{cn_{\pm}}{2} \int d\gamma' d\mu F_e(\gamma') \frac{\beta_{cm} \gamma_{cm}^2}{\gamma \gamma'} \sigma_{total}(\gamma_{cm}). \quad (4.6)$$

The integration is over the region $\gamma_{min} \leq \gamma' \leq \gamma_{max}$ and $-1 \leq \mu \leq 1$ without any restriction. Here the limiting energies of the pairs are denoted by γ_{min} and γ_{max} . Finally, σ_{total} is the total cross section for the pair annihilation, which is obtained by dividing the one given by equation (3.8) with $2\beta_{cm}^2$.

4.3. Bremsstrahlung Cooling Rate

Since this process is much slower than all other reactions (roughly by a factor of α —the fine structure constant) we can treat it to be continuous in the energy and momentum (i.e., $\Delta\gamma/\gamma \ll 1$) and use a continuity equation to describe it. At any time t , the density of electrons in the energy interval $(\gamma, \gamma + d\gamma)$ is given by $n_e F_e(\gamma) d\gamma$. Clearly $n_e F_e(\gamma) \dot{\gamma}(\gamma)$ is the flux density of the electrons entering this interval, and $n_e F_e(\gamma + d\gamma) \dot{\gamma}(\gamma + d\gamma)$ is that due to the electrons leaving this interval (notice that $\dot{\gamma}$ is negative in the case of electron cooling). The net contribution to the electron or positron kinetic equation is now given by

$$\frac{\partial}{\partial t} [n_e(t) F_e(\gamma, t)] = - \frac{\partial}{\partial \gamma} [n_e(t) F_e(\gamma, t) \dot{\gamma}] \equiv C(\gamma, t). \quad (4.7)$$

The right-hand side of this equation is essentially $4\pi\beta\gamma^2(\eta - \chi f)$ for the process. The cooling rate $|\dot{\gamma}|$ can be written as the sum

$$|\dot{\gamma}| = E_{ep}(\gamma) + \int_1^{\infty} d\gamma' F_e(\gamma') [E_{ee}(\gamma, \gamma') + E_{e\bar{e}}(\gamma, \gamma')]. \quad (4.8)$$

The cooling rates E_{ee} , $E_{e\bar{e}}$, and E_{ep} for e^+e^- , e^+e^\pm , and e^\pm -proton processes, respectively, are given in the Appendix (see also Fig. 2).

4.4. The Effect of Coulomb Collisions

Finally, we analyze the effect of Bhabha and Møller collisions (collectively termed as Coulomb collisions) on the

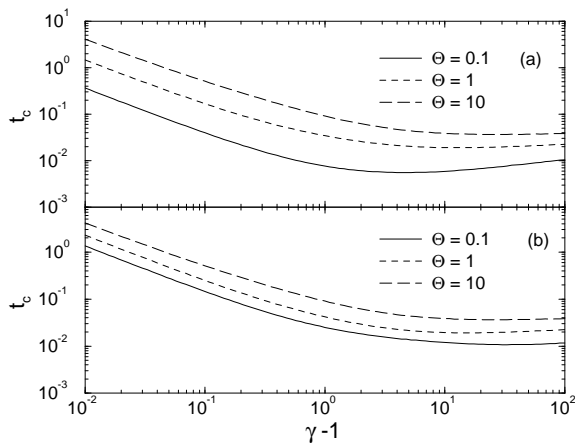


FIG. 2.—Bremsstrahlung cooling time for (a) an electron and (b) a positron of energy γ in a background thermal plasma (of electrons only) of density n_e . In the former case we use the cooling rate E_{ee} and we use $E_{e\bar{e}}$ for the latter. Remaining cooling rates vanish in this particular example. The dimensionless cooling time is defined by $t_c = |\dot{\gamma}| / (cn_e \sigma_{Th} \gamma)$. Since the main time scale in the kinetics of the plasma is $\approx (cn_e \sigma_{Th})^{-1}$, $t_c \ll 1$, at higher energies this means that bremsstrahlung cooling is not very efficient at these energies.

electron spectrum (see e.g., Baring 1987b or CB90 for a similar treatment and Dermer & Liang 1989 for a Fokker-Planck treatment of this problem). Here we ignore the diffusion term (which arises from the second-order derivatives with respect to energy) that would arise in the Fokker-Planck expansion of the kinetic equation as well as the contribution from the pair-proton collisions (which is a much slower process). Consider an elastic scattering in which an electron with momentum p exchanges momentum q with a target particle in the plasma which is either an electron or a positron. In both cases the collision cross section diverges for $|q| \rightarrow 0$, and it falls off rapidly for larger values of $|q|$. We define θ to be the angle by which the incident electron is scattered. Small values of $|q|$ correspond to the small-angle collisions ($\theta \ll 1$). More precisely, $|q| \cong |p|\theta$ when θ is small. It is well known that the divergence of the cross section for $\theta \rightarrow 0$ results in the domination of the relaxation process by the scattering events with small angular deflections. In many situations we can completely ignore the contribution from the collisions which are producing large-angle deflections. Let $l_{\pi/2}$ be the distance an electron has to travel in order that its mean-square deflection is $\cong \pi/2$ and suppose $L_{\pi/2}$ is the distance it has to travel so that it is deflected by an angle of $\pi/2$ in a single scattering, with a probability close to unity. It can be shown that $L_{\pi/2} \cong 16\bar{\gamma}^2 / (45\pi n_e r_e^2)$ and $L_{\pi/2} / l_{\pi/2} \cong 2 \ln \Lambda_C$, where $\bar{\gamma}$ is the mean electron momentum in the background plasma. The latter ratio, in a nonrelativistic plasma, turns out to be $8 \ln \Lambda_C$, but the expression for Λ_C is different in that case. The Coulomb logarithm for a relativistic plasma can be shown to be $\ln \Lambda_C \cong 37 + (3 \ln \bar{\gamma} - \ln n_e)/2$. In this equation n_e refers to the number of electrons per cubic centimeter. We consider only those plasmas for which $\ln \Lambda_C$ is greater than a few, which means that only small-momentum-transfer collisions are relevant. In this limit the Bhabha and Møller cross sections are equal. Therefore, we do not distinguish between electrons and positrons in the foregoing analysis. Consider two distributions f_1 and f_2 of electrons. The Boltzmann equation for f_1 can be written as a continuity equation in the momentum space as

$$\frac{\partial}{\partial t} f_1(p) = - \frac{\partial}{\partial p^i} S_1^i(p), \quad (4.9)$$

where S_1^i is the flux vector in the momentum space (see the Appendix for its definition). Combining equation (2.8) with the above continuity equation, we obtain

$$[\eta(\gamma) - \chi(\gamma)f(\gamma)]_1 = C_{11}(\gamma) + C_{12}(\gamma), \quad (4.10)$$

where

$$C_{1s}(\gamma) = 4\pi^2 c r_e^2 \ln \Lambda_C B \frac{\partial}{\partial \gamma} \int d\gamma' \beta \beta' \gamma'^2 Q(\gamma, \gamma'), \quad (4.11)$$

while

$$Q(\gamma, \gamma') = \left[f_1(\gamma) \frac{\partial}{\partial \gamma'} f_s(\gamma') - f_s(\gamma') \frac{\partial}{\partial \gamma} f_1(\gamma) \right] \int_{-1}^1 d\mu B_0(\gamma, \gamma', \mu). \quad (4.12)$$

The derivation of equation (4.11), along with the definition of the quantities involved, is given in the Appendix. Here C_{11} comes from the collisions within the electrons of distribution f_1 and C_{12} comes from the collisions of electrons of distribution f_1 with the electrons of distribution f_2 . In

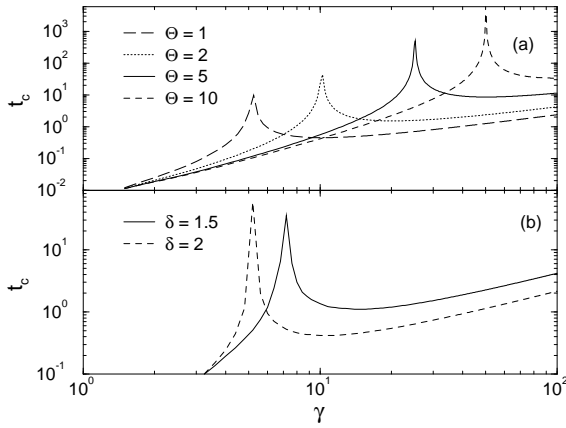


FIG. 3.—Coulomb collision time for (a) a power-law distribution with an index $\delta = 2$ relaxing in a thermal background and (b) a power-law distribution relaxing through self-interactions. In both cases we have used $\ln \Lambda_c = 20$. In general, the time it takes to establish thermal equilibrium is many times that of the collision time. The spikes in these figures indicate that the emission and absorption rates balance at that energy because of the form of C_{1s} (see the text).

each case we have to use the appropriate electron density in the Coulomb logarithm. Clearly, C_{11} vanishes when f_1 is an equilibrium distribution. In Figure 3 we give two examples of Coulomb relaxation. Recall that for electrons $f(\gamma) = nF(\gamma)/(4\pi\beta\gamma^2)$. In the first example (Fig. 3a) we consider a nonthermal population of density n_1 and spectrum $F_1 \propto \gamma^{-2}$ for $\gamma \geq 1$ interacting with a thermal background of density $n_2 \gg n_1$ (f_2 is the background distribution) and temperature Θ . The electrons relax mainly through collisions with the background and the reaction rate is determined by C_{12} above (C_{11} is negligible). The dimensionless collision time is defined by $t_c = cn_2 \sigma_{Th} f_1(\gamma) / |C_{12}(\gamma)|$. In the second example (Fig. 3b) we consider a power-law distribution of density n_1 and an index δ , relaxing through self-interactions (there is no thermal background). In this case $t_c = cn_1 \sigma_{Th} f_1(\gamma) / |C_{11}(\gamma)|$.

5. THE COMPUTATIONAL METHOD

In the kinetic theory approach to nonequilibrium plasmas that we have presented in the preceding sections, the computational task is reduced to evaluating many collision integrals (for each energy bin, after each time step) quickly and efficiently, without compromising on the flexibility to handle many types of distribution functions. Now we explain our new algorithm for adaptive Monte Carlo integration which meets this demand. Our approach is similar to the PSD method discussed in the introduction, with the principal difference being that we are not using the conventional MC method (based on uniform sampling) to compute the transition probabilities (the collision integrals). There are also some similarities between our method and the LP method described in the introduction. Both methods use statistical weights within a Monte Carlo scheme. In the LP method, these weights are introduced in an ad hoc fashion, based on the energy carried by the LPs. In our method, we use probability weights (see below) to enhance the sampling rate in those regions where the contribution to the integral being evaluated is greater. But these weights (known as importance weights) are generated internally, through a minimal-variance prescription (see eqs. [5.12]

and [5.14]). Therefore, what we are using is a Monte Carlo method based on importance sampling.

Being a kinetic theory approach, our method bears a lot of resemblance to that of C92. Both methods discretize the kinetic equations by following the particle and photon statistics with reference to energy bins which are equally spaced logarithmically. Both methods have similar simplifying assumptions about the isotropy and the spatial uniformity of the photon and pair distribution functions. Our method is more flexible than that of C92 because of the adaptive nature of the underlying Monte Carlo scheme (C92 uses numerical integration to compute the collision kernels). However, there are several differences between these two methods. It is straightforward to deal with anisotropic distributions in our method, except that some of the collision integrals given in this paper require some modifications to incorporate them. Moreover, we have not taken into account, the escape of photons and pairs from the system. We use the exact collision kernels throughout. In contrast to the method of C92, our time-evolution code is entirely dynamic, i.e., we do not precompute and store any quantities, which makes our method more flexible. The integrands of some of the collision kernels have very narrow peaks (“integrable singularities”) which are hard to evaluate through numerical methods used before (e.g., C92 and references therein). Such “singularities” can be easily integrated by our method. No special attention is required because the algorithm is adaptive; it automatically adjusts the sampling rate, iteratively, to a high value at such points. Now we describe our computational method in detail.

5.1. Discretization of the Kinetic Equations

In order to simplify the analysis, we will consider only the case for which $n_+ = n_- = n_e$ and $F_+(\gamma) = F_-(\gamma) = F_e(\gamma)$. This can be easily extended to the more general case. Let the net collision rate for the photons, due to Compton scattering, be given by

$$A_\gamma(\epsilon, t) = 4\pi\epsilon^2 [\eta_\gamma(\epsilon, t) - f_\gamma(\epsilon, t)\chi_\gamma(e, t)]_{(e\gamma \leftrightarrow e\gamma)}. \quad (5.1)$$

The corresponding collision rate due to pair annihilation and creation ($ee \rightarrow \gamma\gamma$) is denoted by $B_\gamma(\epsilon, t)$. In an analogous way we define $A_e(\gamma, t)$ and $B_e(\gamma, t)$ for the corresponding collision rates for pairs. From the photon rate equation

$$\frac{\partial}{\partial t} [n_\gamma(t)F_\gamma(\epsilon, t)] = A_\gamma(\epsilon, t) + B_\gamma(\epsilon, t), \quad (5.2)$$

we obtain

$$\Delta F_\gamma(\epsilon, t) = \frac{[A_\gamma(\epsilon, t) + B_\gamma(\epsilon, t)]\Delta t - F_\gamma(e, t)\Delta n_\gamma(t)}{n_\gamma(t) + \Delta n_\gamma(t)}, \quad (5.3)$$

where $\Delta F_\gamma(\epsilon, t) = F_\gamma(\epsilon, t + \Delta t) - F_\gamma(\epsilon, t)$, $\Delta n_\gamma(t) = \dot{n}_\gamma(t)\Delta t$, and $\dot{n}_\gamma(t)$ is given below. Similarly, we can obtain $\Delta F_e(\gamma, t)$. The time increment Δt for each time step is chosen in such a way that

$$\left| \frac{\partial}{\partial t} f_\gamma(\epsilon, t) \right| \Delta t \leq v f_\gamma(\epsilon, t) \quad \text{and} \quad \left| \frac{\partial}{\partial t} f_e(\gamma, t) \right| \Delta t \leq f_e(\gamma, t) \quad (5.4)$$

for all values of ϵ and γ . In our computation we have used $v = 0.1$, which means that the maximum change in F_e or F_γ in any energy bin, during any time step, is less than or equal to 10%. Now we determine \dot{n}_γ arising from the pair pro-

cesses (there is no change in n_e or n_γ arising from Compton scattering). We have

$$\int_0^\infty d\epsilon A_\gamma(\epsilon, t) = 0 \quad \text{and} \quad \dot{n}_\gamma(t) = \int_0^\infty d\epsilon B_\gamma(\epsilon, t) \quad (5.5)$$

and two analogous equations for pairs. It can be shown that the positron annihilation and creation rates are given by

$$\dot{n}_{\text{ann}}(t) = \frac{cn_e^2(t)}{2} \int d\mu d\gamma d\gamma' F_e(\gamma, t) F_e(\gamma', t) \frac{\beta_{\text{cm}} \gamma_{\text{cm}}^2}{\gamma\gamma'} \sigma_{ee \rightarrow \gamma\gamma} \quad (5.6)$$

and

$$\dot{n}_{\text{cr}}(t) = \frac{cn_\gamma^2(t)}{8} \int d\mu d\epsilon d\epsilon' F_\gamma(\epsilon, t) F_\gamma(\epsilon', t) (1 - \mu) \sigma_{\gamma\gamma \rightarrow ee}, \quad (5.7)$$

respectively. Here $\sigma_{ee \rightarrow \gamma\gamma}$ and $\sigma_{\gamma\gamma \rightarrow ee}$ are the total cross sections, which are given in the previous sections. For each positron annihilated or created, there will be a creation or annihilation, respectively, of two photons. Therefore we have

$$\dot{n}_\gamma(t) = 2[\dot{n}_{\text{ann}}(t) - \dot{n}_{\text{cr}}(t)] \quad \text{and} \quad \dot{n}_e(t) = \dot{n}_{\text{cr}}(t) - \dot{n}_{\text{ann}}(t). \quad (5.8)$$

We remark that $\dot{n}_- = \dot{n}_+ = \dot{n}_e$. We have verified equations (5.5) and (5.8) are satisfied in all our computations for time evolution, which implies that the particle number is conserved. In addition, we have verified the conservation of the total energy after each time step. Now we can use equation (5.3) iteratively, to obtain the time evolution of F_e and F_γ from the initial data, viz., $F_e(\gamma, 0)$, $F_\gamma(\epsilon, 0)$, and the initial densities. We have discretized the energy (ϵ and γ) with twenty energy bins per decade of energy and used a logarithmic interpolation between these points to reconstruct F_e and F_γ for the subsequent time steps, which are then used in the collision integrals. Now the problem reduces to an efficient evaluation of these multidimensional collision integrals with complicated integrands. For this purpose we have developed a new version of an adaptive and iterative Monte Carlo method. It progressively adjusts itself to the nature of the integrand. We describe our algorithm below.

5.2. The Adaptive Monte Carlo Method

A general purpose algorithm for multidimensional integration which is widely used in the experimental particle physics is given by Lepage (1978). It is an iterative and adaptive scheme. A computer program implementing this method, known as VEGAS, can be found in Press et al. (1992). However, we have found that it has several shortcomings when applied to the type of integrals that arise in the kinetic theory. Not only is the convergence weak in some cases, we have found that the subroutine gave erroneous output for the high-energy tails of the distributions. This is a significant obstacle because of the integrals over energy that we have to perform at the end of each time step. That integration makes the errors propagate to lower energies (where the results are otherwise accurate) during the succeeding time steps. We will briefly explain the original method by Lepage and then describe our modified scheme which can handle the integrals we need. First, by scaling the

integration variable, any multidimensional integral can be written in the form

$$\mathcal{J} = \int dr f(r), \quad (5.9)$$

where $r = (z_1, z_2, \dots, z_n)$, $dr = \prod_{i=1}^n dz_i$, f is the function to be integrated (which is continuous and well behaved) and the integration is over the n -dimensional hypercube $0 \leq z_i < 1$, $i = 1, 2, \dots, n$. If we generate $M \gg 1$ random points r_k with a normalized probability density $p(r)$ then the integral can be approximated by

$$\mathcal{J} \simeq \frac{1}{M} \sum_k \frac{f(r_k)}{p(r_k)}. \quad (5.10)$$

The variance is given by

$$\begin{aligned} \sigma^2[p] &= \frac{1}{M-1} \left[\int dr \frac{f^2(r)}{p(r)} - \mathcal{J}^2 \right] \\ &\rightarrow \frac{1}{M-1} \left[\sum_k \frac{f^2(r_k)}{p^2(r_k)} - \mathcal{J}^2 \right]. \end{aligned} \quad (5.11)$$

The optimal choice for $p(r)$ which minimizes the variance is derived from

$$\frac{\delta}{\delta p} \left\{ \sigma^2[p] + \lambda \int dr p(r) \right\} = 0, \quad (5.12)$$

which implies that

$$p(r) = \frac{|f(r)|}{\mathcal{J}}. \quad (5.13)$$

This is the central theme of the importance sampling technique, sample more in the regions where the absolute value of the function is larger. However, observe that the denominator is the integral itself! Thus we need an algorithm to solve it iteratively, starting with a reasonable guess for p . Then we calculate the integral by using equation (5.10) which then determines the new form for $p(r)$, and so on. If this process converges in a manageable number of iterations, then we will have achieved our goal. The data storage requirements of directly implementing this scheme are well within the reach of many present-day computers. The method by Lepage consists of a restrictive assumption that the probability density is separable. For instance, when $n = 2$ and $r = (x, y)$, the separability means that $p(x, y) = p_x(x)p_y(y)$ and to minimize the variance we need

$$\frac{\delta}{\delta p_x} \left\{ \sigma^2[p_x, p_y] + \lambda_x \int_0^1 dx p_x(x) + \lambda_y \int_0^1 dy p_y(y) \right\} = 0, \quad (5.14)$$

which implies that

$$p_x(x) = \frac{\left\{ \int_0^1 dy [f^2(x, y)/p_y(y)] \right\}^{1/2}}{\int_0^1 dx \left\{ \int_0^1 dy [f^2(x, y)/p_y(y)] \right\}^{1/2}}, \quad (5.15)$$

and a similar equation for $p_y(y)$. For arbitrary dimensions, this scheme is implemented in the VEGAS subroutine, mentioned before. The motivation for assuming the separability, according to Lepage (1978), is that it limits the storage requirements. It is not a good assumption in general. Therefore we proceed to implement importance sampling directly. All essential features of the algorithm can be captured in a one-dimensional example which we will consider

first. Then we will show how it can be generalized to higher dimensions. Consider the integral $\mathcal{J} = \int_0^1 dx f(x)$. Let $p(x)$ be the normalized probability density we want. Suppose N is an integer greater than unity and $0 = x_0 < x_1 < x_2 < \dots < x_N = 1$, while $\Delta x_i = x_i - x_{i-1}$ for $i = 1, 2, \dots, N$. We will use the following discrete representation of the probability density:

$$p(x) = \frac{1}{N\Delta x_i} \quad \text{if } x_{i-1} \leq x < x_i, \quad (5.16)$$

so that $\int_{x_{i-1}}^{x_i} dx p(x) = 1/N$ for all i . Here the bin sizes Δx_i need not be all equal, but all bins have the same probability weight. If the bin sizes are equal, we will get a uniform probability distribution leading to the crude Monte Carlo method. Now the integral is approximated by $\mathcal{J} = \sum_{k=1}^M f(a_k)/Mp(a_k)$, where $0 \leq a_k < 1$ are uniformly distributed random numbers. Typically $M \gg N$. Let

$$u_i = \frac{N}{M} \sum_{k=1}^M c_i(k) |f(a_k)|, \quad (5.17)$$

where $c_i(k) = 1$, if $x_{i-1} \leq a_k < x_i$, and is zero otherwise. Clearly, $\sum_{i=1}^N u_i \Delta x_i = \mathcal{J}$. Therefore $w_i = u_i \Delta x_i / \mathcal{J}$ is the importance weight associated with the i th bin. Since different bins contribute different amounts to the integral, the idea now is to find a new set of bin spacings $\{x_1, x_2, \dots, x_{N-1}\}$, so that all bins have equal importance weight $w_0 = 1/N$. Let l be an integer (which depends on the bin location i) such that

$$\sum_{m=1}^l w_m \leq iw_0 < \sum_{m=1}^{l+1} w_m. \quad (5.18)$$

Then the new grid position for the i th bin can be obtained from

$$x_{i,\text{new}} = x_{l,\text{old}} + \frac{1}{w_0} \left(iw_0 - \sum_{m=1}^l w_m \right) (x_{l+1,\text{old}} - x_{l,\text{old}}). \quad (5.19)$$

However, in practice we must damp the convergence so that the contribution from the low-importance bins is not overly suppressed. As in the method by Lepage, we will damp the convergence by using the modified importance weights given by

$$w'_i = \left[\frac{1 - w_i}{\log(1/w_i)} \right]^\alpha, \quad (5.20)$$

which gives $w'_0 = \sum_{i=1}^N w'_i / N$. We now replace w_0 and w_i with the corresponding primed quantities in the above equations. The new probability density is now determined by using equation (5.16) and the process is repeated iteratively. If it converges, we will have $x_{i,\text{new}} \cong x_{i,\text{old}}$ for all i , from which we can obtain the desired estimate for \mathcal{J} . Now we give the extension of this scheme to two dimensions. We will assume that the number of bins is N for each dimension. A discrete representation of the probability density is given by

$$p(x, y) = \frac{1}{N^2 \Delta x_i \Delta y_j} \quad (5.21)$$

if $x_{i-1} \leq x < x_i$ and $y_{j-1} \leq y < y_j$. This does not mean that the probability density is separable because Δx_i and Δy_j are not independent in general. The integral is now estimated

by

$$\mathcal{J} \simeq \frac{1}{M} \sum_{k=1}^M \frac{f(a_k, b_k)}{p(a_k, b_k)}, \quad (5.22)$$

where $0 \leq a_k < 1$ and $0 \leq b_k < 1$ are uniformly distributed random numbers. Let

$$h_{ij} = \frac{N^2}{M} \sum_{k=1}^M c_{ij}(k) |f(a_k, b_k)|, \quad (5.23)$$

where $c_{ij}(k) = 1$, if $x_{i-1} \leq a_k < x_i$ and $y_{j-1} \leq b_k < y_j$, and is zero otherwise. Now we define $u_i = \sum_{j=1}^N h_{ij} \Delta y_j$ and $v_j = \sum_{i=1}^N h_{ij} \Delta x_i$. Clearly, $\mathcal{J} = \sum_{i=1}^N u_i \Delta x_i = \sum_{j=1}^N v_j \Delta y_j$. Let $w_{x_i} = u_i \Delta x_i / \mathcal{J}$ and $w_{y_j} = v_j \Delta y_j / \mathcal{J}$. From these importance weights for x and y grids we can obtain the corresponding damped weights and proceed to iterate as if these were two one-dimensional problems. Generalization to arbitrary dimensions is now straightforward.

In all our applications we found that the values $N = 70$ and $\alpha = 1.3$ (for the damping index) gave stable and satisfactory results within at most 10 iterations or so. In general it is advisable to start with a few thousand samples and after several iterations, increase M (and retaining the resulting grid) and further iterate, and so on. For many types of integrals, of at most five dimensions, we found that $M_{\text{max}} \simeq 10^4$ samples to be adequate. For all the results presented in this paper, we have used the subroutine ran2 in Press et al. (1992) for the random number generation. We find that our method is faster than the crude Monte Carlo method (using uniform sampling) by a factor of 10 or better, which is also the case with the method by Lepage (when it is applicable).

6. TIME EVOLUTION AND EQUILIBRIA

Here we give an analytical description of the equilibrium states of a pair plasma, in terms of the initial conditions. For two specific examples, we follow the relaxation toward equilibrium using our time-evolution code. These examples are meant to demonstrate that the whole formalism of this paper (the collision integrals and the computational method) actually works. We are considering a homogeneous, stationary, isotropic, and nonmagnetic system. There are no radiative transfer or hydrodynamic effects. On short timescales $t \approx t_{\text{Th}} = (n_+ \sigma_{\text{Th}} c)^{-1}$ the kinetics is determined by the rate equations alone (see eq. [2.8]). We have seen that the collision integrals for these equations are nonlinear functionals of the distribution functions. Given the initial state of the plasma, we can solve these first-order coupled and nonlinear integro-differential equations to determine the time evolution of the distributions. The system is characterized by the densities n_γ , n_+ , and n_- and the spectra $F_\gamma(\epsilon)$ and $F_e(\gamma)$, all of which depend on time. Their values at $t = 0$ define the initial state of the system. The total density of the particles is given by $\tilde{n} = n_\gamma + 2n_+ + n_-$, and the total energy density (including the rest energy of the pairs) is given by $\tilde{u} = u_\gamma + u_- + u_+$, where $u_\gamma = n_\gamma \int_0^\infty \epsilon F_\gamma(\epsilon) d\epsilon$ and $u_\pm = n_\pm \int_1^\infty \gamma F_e(\gamma) d\gamma$. The mean energy per particle is given by $\bar{\epsilon} = \tilde{u} / \tilde{n}$. We see that there will be no change in \tilde{n} due to Compton scattering or the pair annihilation and creation. It will change only as a result of bremsstrahlung (also double Compton scattering and the pair annihilation into three photons) which operates on a longer timescale $t \approx t_{\text{Th}} / \alpha$ (α is the fine structure constant). However, \tilde{u} remains constant throughout. Therefore we can divide the approach of the system toward equilibrium into two phases: (1) The faster

phase in which both \tilde{u} and \tilde{n} remain constant and the system approaches to a state of kinetic equilibrium so that the total reaction rates for Compton scattering and the pair annihilation vanish (separately). This state is characterized by a temperature $\tilde{\Theta}$ and the chemical potentials $\tilde{\mu}_\gamma$ and $\tilde{\mu}_\pm$. (2) The slower phase in which \tilde{u} is constant but \tilde{n} changes, mainly due to bremsstrahlung (or its inverse, and other radiative processes) so that the system finally reaches a thermal equilibrium state characterized by a temperature Θ_0 and a total density n_0 . In this state the chemical potentials vanish (see below). If $\Theta_0 < \tilde{\Theta}$ then $n_0 > \tilde{n}$, which means that this phase is mainly the cooling of the plasma through bremsstrahlung and other similar processes. On the other hand, if $\Theta_0 > \tilde{\Theta}$ then the plasma will heat up due to the inverse bremsstrahlung (free-free absorption) and other radiative processes.

6.1. Kinetic Equilibrium: The Densities and the Temperature

Consider Compton scattering of an electron of energy γ and a photon of energy ϵ . The respective energies after the scattering are taken to be γ' and ϵ' . If the total reaction rate vanishes, then we have

$$f(\gamma)f_\gamma(\epsilon)\left[1 + \frac{\lambda_0^3}{2}f_\gamma(\epsilon')\right] = f(\gamma')f_\gamma(\epsilon')\left[1 + \frac{\lambda_0^3}{2}f_\gamma(\epsilon)\right], \quad (6.1)$$

where we have retained the Bose-Einstein enhancement factor for the photons and $\lambda_0 = h/mc$. The factor half in this equation takes into account the polarization degeneracy of the photon states. Using the general form of the distribution functions

$$f_\gamma(\epsilon) = \frac{2}{\lambda_0^3[\exp(\epsilon - \mu_\gamma)/\Theta_\gamma - 1]} \quad (6.2)$$

and

$$f_\pm(\gamma) = \frac{2}{\lambda_0^3} \exp\left(\frac{\mu_\pm - \gamma}{\Theta_\pm}\right)$$

and equation (6.1), we find $\Theta_+ = \Theta_\gamma = \Theta_-$. We denote this common temperature by $\tilde{\Theta}$. Notice that equation (6.1) does not yield any condition on the chemical potentials. Now requiring that the total reaction rate should vanish for the pair annihilation and creation as well, we find

$$f_+(\gamma_+)f_-(\gamma_-)\left[1 + \frac{\lambda_0^3}{2}f_\gamma(\epsilon_1)\right]\left[1 + \frac{\lambda_0^3}{2}f_\gamma(\epsilon_2)\right] = f_\gamma(\epsilon_1)f_\gamma(\epsilon_2), \quad (6.3)$$

where γ_\pm are the pair energies and $\epsilon_{1,2}$ are the photon energies. Using the fact that the pairs and the photons have a common temperature $\tilde{\Theta}$, we obtain from this equation $\tilde{\mu}_- + \tilde{\mu}_+ = 2\tilde{\mu}_\gamma$. If there are no ions in the plasma (i.e., $n_p = 0$) then $\tilde{n}_- = \tilde{n}_+$ so that $\tilde{\mu}_- = \tilde{\mu}_+ = \tilde{\mu}_\gamma$. By assuming that $\exp[(\epsilon - \tilde{\mu}_\gamma)/\tilde{\Theta}] \gg 1$ and $\exp[(\gamma - \tilde{\mu}_\pm)/\tilde{\Theta}] \gg 1$, for the relevant energies, we obtain the distribution functions in the kinetic equilibrium state to be

$$f_\gamma(\epsilon) = \frac{2}{\lambda_0^3} \exp\left(\frac{\tilde{\mu}_\gamma - \epsilon}{\tilde{\Theta}}\right) \quad \text{and} \quad f_\pm(\gamma) = \frac{2}{\lambda_0^3} \exp\left(\frac{\tilde{\mu}_\pm - \gamma}{\tilde{\Theta}}\right). \quad (6.4)$$

The densities are given by

$$\tilde{n}_\gamma = \int_0^\infty 4\pi\epsilon^2 f_\gamma(\epsilon) d\epsilon = 16\pi\left(\frac{\tilde{\Theta}}{\lambda_0}\right)^3 \exp\left(\frac{\tilde{\mu}_\gamma}{\tilde{\Theta}}\right) \quad (6.5)$$

and

$$\tilde{n}_\pm = \int_1^\infty 4\pi\gamma\sqrt{\gamma^2 - 1} f_\pm(\gamma) d\gamma = \frac{8\pi}{\lambda_0^3} \tilde{\Theta} K_2\left(\frac{1}{\tilde{\Theta}}\right) \exp\left(\frac{\tilde{\mu}_\pm}{\tilde{\Theta}}\right), \quad (6.6)$$

where K_n is the n th-order modified Bessel function of the second kind. Using the relation $2\tilde{\mu}_{-\gamma} = \tilde{\mu}_- + \tilde{\mu}_+$ we find

$$\tilde{n}_\gamma = 4\zeta^2 \tilde{n}_+(n_p + \tilde{n}_+), \quad (6.7)$$

where $\zeta = \tilde{\Theta}^2/K_2(1/\tilde{\Theta})$. Finally, from the equation $\tilde{n}_\gamma + 2\tilde{n}_+ + n_p = \tilde{n}$, we obtain the densities \tilde{n}_γ and \tilde{n}_+ in terms of \tilde{n} and n_p . When $\zeta \neq 1$ we obtain a quadratic equation for \tilde{n}_+ . It turns out that only one of its roots is physical (i.e., both \tilde{n}_γ and \tilde{n}_+ are nonnegative). The physical root is given by

$$\tilde{n}_+ = \frac{1}{2}[(\tilde{n} - n_*)(1 - \zeta^2)^{-1} - n_p], \quad (6.8)$$

where $n_* = \zeta[\tilde{n}^2 - (1 - \zeta^2)n_p^2]^{1/2}$. When $\zeta = 1$ (which is true when $\tilde{\Theta} = 0.493$) we get $\tilde{n}_+ = (\tilde{n} - n_p)^2/(4\tilde{n})$. Therefore we have the necessary densities in terms of the temperature. When there are no ions ($n_p = 0$) these solutions take a simple form given by

$$\tilde{n}_- = \tilde{n}_+ = \frac{K_2(1/\tilde{\Theta})}{2[\tilde{\Theta}^2 + K_2(1/\tilde{\Theta})]} \tilde{n} \quad (6.9)$$

and

$$\tilde{n}_\gamma = \frac{\tilde{\Theta}^2}{\tilde{\Theta}^2 + K_2(1/\tilde{\Theta})} \tilde{n}.$$

Now we determine the temperature in terms of the initial data. We have

$$\tilde{\mu}_\gamma = \int_0^\infty 4\pi\epsilon^3 f_\gamma(\epsilon) d\epsilon = 3\tilde{\Theta}\tilde{n}_\gamma \quad (6.10)$$

and

$$\tilde{\mu}_\pm = \int_1^\infty \gamma^2 \sqrt{\gamma^2 - 1} f_\pm(\gamma) d\gamma = \frac{3\tilde{\Theta}K_2(1/\tilde{\Theta}) + K_1(1/\tilde{\Theta})}{K_2(1/\tilde{\Theta})} \tilde{n}_\pm. \quad (6.11)$$

Using the energy conservation equation $\tilde{\mu} = \tilde{\mu}_\gamma + \tilde{\mu}_- + \tilde{\mu}_+$, we get the temperature as an implicit function of $\tilde{\mu}$, \tilde{n} , and n_p . In the limit where $n_p = 0$, we have

$$\tilde{\mu}_\gamma = \frac{3\tilde{\Theta}^3}{\tilde{\Theta}^2 + K_2(1/\tilde{\Theta})} \tilde{n} \quad (6.12)$$

and

$$\tilde{\mu}_- + \tilde{\mu}_+ = \frac{3\tilde{\Theta}K_2(1/\tilde{\Theta}) + K_1(1/\tilde{\Theta})}{\tilde{\Theta}^2 + K_2(1/\tilde{\Theta})} \tilde{n}.$$

In this case, the equation for the temperature takes the form

$$3\tilde{\Theta}^3 + 3\tilde{\Theta}K_2(1/\tilde{\Theta}) + K_1(1/\tilde{\Theta}) = \bar{\epsilon}[\tilde{\Theta}^2 + K_2(1/\tilde{\Theta})], \quad (6.13)$$

where $\bar{\epsilon}$ is the mean energy per particle (which is determined by the initial conditions).

6.2. Thermal Equilibrium: Densities and the Temperature

Here we determine the final temperature and densities resulting from the radiative processes in the second phase. We have $\mu_- + \mu_+ = 2\mu_\gamma = 0$. Let $\mu_+ = -\mu_- = \mu_0$ and $z = \exp(\mu_0/\Theta_0)$. Clearly

$$n_{\pm} = \frac{8\pi}{\lambda_0^3} \Theta_0 K_2(1/\Theta_0) z^{\pm 1} \quad \text{and} \quad n_\gamma = \frac{16\pi\Theta_0^3}{\lambda_0^3}. \quad (6.14)$$

Using the fact that $n_- = n_p + n_+$, we can show that

$$n_- + n_+ = \frac{16\pi}{\lambda_0^3} \Theta_0 K_2(1/\Theta_0) \sqrt{1+x^2}, \quad (6.15)$$

where $x = \lambda_0^3 n_p / [16\pi\Theta_0 K_2(1/\Theta_0)]$. In the nonrelativistic limit ($\Theta_0 \ll 1$) the pair density is given by

$$n_- + n_+ = \frac{4}{\lambda_0^3} (2\pi\Theta_0)^{3/2} \exp(-1/\Theta_0) \times \left[1 + \frac{15}{8} \Theta_0 + \frac{105}{128} \Theta_0^2 \right] \sqrt{1+x^2}. \quad (6.16)$$

It can be shown that the pair energy density is given by

$$u_- + u_+ = \frac{16\pi}{\lambda_0^3} [3\Theta_0^2 K_2(1/\Theta_0) + \Theta_0 K_1(1/\Theta_0)] \sqrt{1+x^2}. \quad (6.17)$$

Finally, energy conservation implies

$$\frac{8\pi^5\Theta_0^4}{15} + 16\pi[3\Theta_0^2 K_2(1/\Theta_0) + \Theta_0 K_1(1/\Theta_0)] \sqrt{1+x^2} = \lambda_0^3 \tilde{\mu}, \quad (6.18)$$

where the first term on the left-hand side is the contribution from the photons. We can solve this equation for Θ_0 in terms of $\tilde{\mu}$ and x (equivalently n_p). This completes the analytical description of the thermal equilibrium state in terms of the initial data. This treatment is exact and is valid for all energies (relativistic or otherwise) and densities (so long as the plasma is nondegenerate).

6.3. Time Evolution of the Spectra: Two Examples

Now we consider the time evolution of the plasma for two specific initial conditions. In the first case the initial photon and the pair distributions are flat (i.e., F is constant) and nonzero within the energy (in MeV) interval $0.1 \leq \epsilon mc^2 \leq 10$ and $0.1 \leq (\gamma - 1)mc^2 \leq 10$. The initial densities are taken to be $n_\gamma = n_+ + n_- = 2 \times 10^{20} \text{ cm}^{-3}$. For this case we find a kinetic-equilibrium temperature $\tilde{\Theta} = 3.43$, and the corresponding densities are found to be $\tilde{n}_{\text{ph}} = 1.36 \times 10^{20} \text{ cm}^{-3}$ and $\tilde{n}_- = \tilde{n}_+ = 1.32 \times 10^{20} \text{ cm}^{-3}$. Monte Carlo evolution of the spectra for this case are shown in Figures 4 and 5. They agree well with the analytical kinetic-equilibrium solutions. For this case, as well as the second one, we have used $t_{\text{Th}} = (c\sigma_{\text{Th}} n)^{-1}$, with $n = 2 \times 10^{20} \text{ cm}^{-3}$. In the second case we start with the same densities of the photons and pairs, and the initial distributions are confined to the same band width as above. The only difference is that $F_\gamma(\epsilon) \propto \epsilon^{-2}$ and $F_e(\gamma) \propto \gamma^{-2}$, with suitable normalizations. In this case we obtain a kinetic-equilibrium temperature $\tilde{\Theta} = 0.663$ and the corresponding densities are found to be $\tilde{n}_{\text{ph}} = 1.73 \times 10^{20} \text{ cm}^{-3}$ and $\tilde{n}_- = \tilde{n}_+ = 1.1 \times 10^{20} \text{ cm}^{-3}$. Monte Carlo spectra for this case

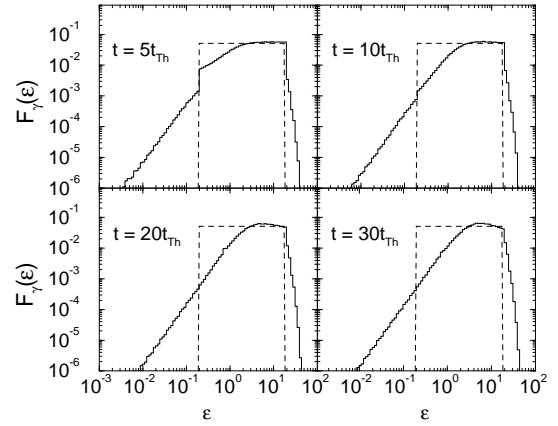


FIG. 4.—Time evolution of the photon spectrum (solid line) starting from a flat initial spectrum (dashed line). Initial pair spectrum is flat as well. It is clear that the softer end of the spectrum relaxes first. The same phenomenon is observed in the pair distribution (not shown here).

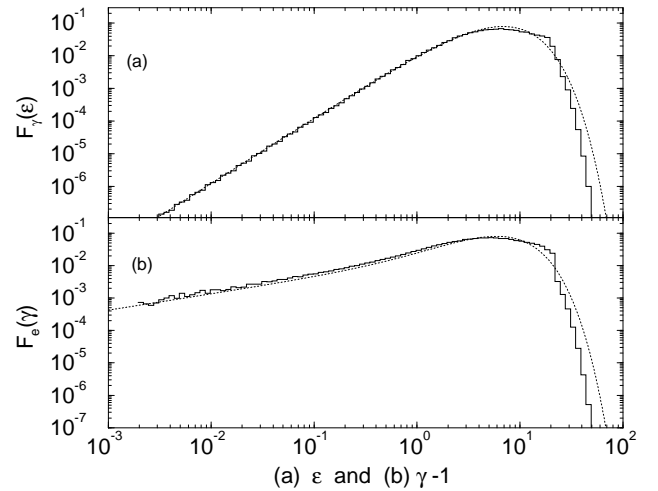


FIG. 5.—Final Monte Carlo spectra (the solid histograms) at $t = 45t_{\text{Th}}$ compared with the analytical solution (dashed curves) for (a) the photons and (b) the pairs, starting from the flat initial spectra. See § 6.3 for details.

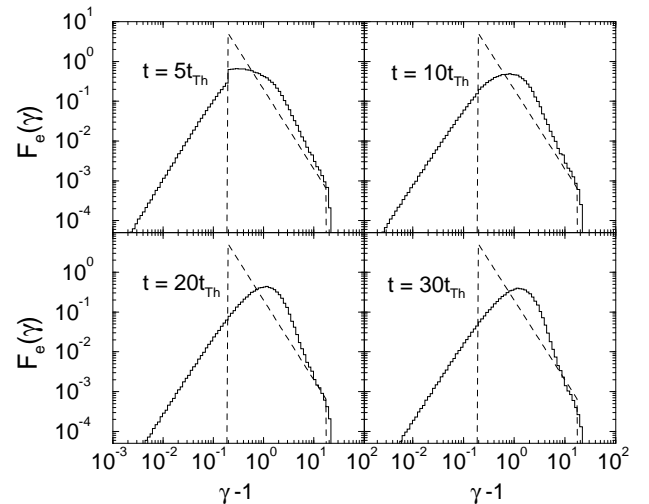


FIG. 6.—Evolution of the photon spectrum (solid line) starting from the power-law ($\delta = 2$) distributions of the photons (dashed line) and the pairs (evolution not shown here). As in the previous example, the relaxation is faster at lower energies.

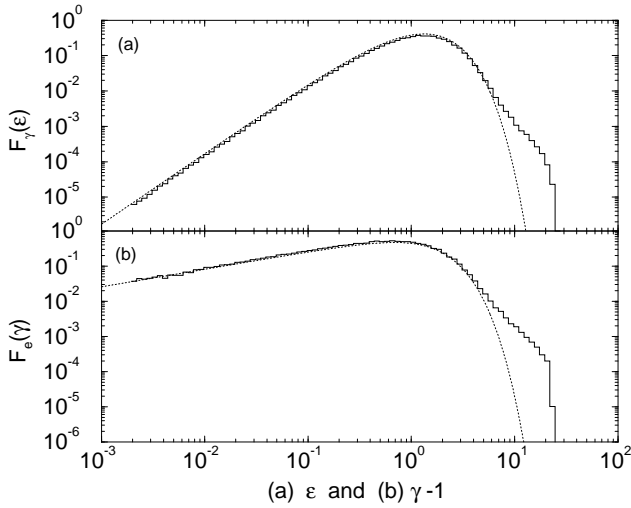


FIG. 7.—Final Monte Carlo spectra (solid histograms) at $t = 40\tau_{th}$ compared with the analytical solution (dashed curves) for (a) the photons and (b) the pairs, starting from the power-law initial spectra. It is evident that the high-energy tails persist for a long time, becoming steeper with time (analogous to the relaxation in a nonrelativistic plasma), but the number of particles (and the energy) in these tails is less than a few percent of the total.

are shown in Figures 6 and 7. Once again they are in a good agreement with the analytical solution. We have verified the number and energy conservation after each time step. The final densities are found to agree with the predicted values within an accuracy of 10% or better (which can be improved by using more energy bins). In both cases the kinetic-equilibrium solution is moderately relativistic. It is clear from Figures 5 and 7 that the low-energy part of the spectrum relaxes before the high-energy end. The cross sections (and hence the reaction rates) decrease with the energy, thereby making the relaxation slower at higher energies. A part of the deviation from the analytical solutions that we see in Figures 5 and 7 (in the high-energy tails) could be an artifact of our sparse (logarithmic) binning at higher energies. It can be rectified by using more bins in the high-energy end (and more computing time). For the above cases the final thermal-equilibrium temperatures turn out to be $\Theta_0 = 4.36 \times 10^{-3}$ and $\Theta_0 = 2.98 \times 10^{-3}$, respectively.

7. CONCLUSIONS

We have developed a new computational method for solving the Boltzmann equations of a pair plasma which is applicable for arbitrary energies (in the X-ray and γ -ray bands), densities, and distribution functions. We have fully

analyzed all relevant microscopic processes in a pair plasma viz., Comptonization, the pair creation and annihilation, bremsstrahlung and the associated cooling, and Coulomb collisions. The spectra from the individual collision integrals, using our expressions and the numerical method (for Compton scattering, pair annihilation, and bremsstrahlung), are in a good agreement with several previous results obtained by using different methods (e.g., S82a; CB90; Dermer 1986). The analysis given in this paper can be very easily extended to an inhomogeneous and anisotropic plasma. It will only change some of the collision integrals and add a spatial component to the kinetic equations. That will result in an increase in the computational time but it will still be manageable by the present-day work stations. Presence of the magnetic fields will alter the kinetics (through synchrotron emission), and it can be modeled along the same lines as that of C92. We have developed a modified version of the adaptive Monte Carlo method which is very efficient and robust. It is faster than the crude Monte Carlo method (using uniform sampling) by at least a factor of 10 and is more flexible than the numerical integration methods (which do not use random sampling) which are used in the past. We have obtained the analytical equilibrium solutions for a general set of initial conditions. Finally, we have tested our Monte Carlo evolution scheme for two specific sets of initial conditions and found that the results compared favorably with the corresponding analytical solutions. The method is found to be very stable. In each of the examples considered, the program has analyzed a total of $\sim 10^{10}$ collision events. This stability, accompanied by its generality and the inherent flexibility, makes this technique suitable for many astrophysical applications. In particular, this formalism can be applied to the expanding pair plasmas in the γ -ray-burst sources in their final stages of evolution (when they are only moderately optically thin), AGN, and the emission from hot accretion discs near black holes.

R. P. is very pleased to thank Marc Kamionkowski for numerous discussions and insightful suggestions. He is happy to thank Malvin Ruderman for many discussions, help, and encouragement. He thanks Alexandre Refregier and Abraham Loeb for interesting discussions and Andrej Zdziarski for sending his papers on this subject. Thanks are due to the referee, Paolo Coppi, for many useful suggestions which made this paper much more concise and clear. This work was partially supported by a grant from NASA, NAG 5-2841.

APPENDIX

A.1. COMPTON SCATTERING RATE FOR PHOTONS

The cross section in the C frame is given by

$$\frac{d\sigma}{dP} = \frac{1}{\epsilon^2} \frac{d\sigma}{d\Omega} \delta(\epsilon - \tilde{\epsilon}), \quad (A1)$$

with $\delta(\epsilon - \tilde{\epsilon}) = \delta(\epsilon_1 - \tilde{\epsilon}_1) d\tilde{\epsilon}_1/d\epsilon$. Here $d\Omega$ is an infinitesimal solid angle around the direction \mathbf{k} (similarly $d\Omega'$ is defined with respect to \mathbf{k}'). It is easy to see that $d\tilde{\epsilon}_1/d\epsilon = \xi^2 a a_1^{-1}$. Now

$$\left(\frac{d\sigma}{d\Omega} \right)_{C\text{-frame}} = \frac{d\Omega'}{d\Omega} \left(\frac{d\sigma}{d\Omega} \right)_{R\text{-frame}}. \quad (A2)$$

Since $\epsilon^2 d\Omega = \epsilon'^2 d\Omega'$ we get $d\Omega/d\Omega' = (\gamma a)^2$. Finally,

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{R-frame}} = \frac{r_e^2 \Delta}{2\xi^3} \quad (\text{A3})$$

is the Klein-Nishina formula in our notation, where $\Delta = \xi^2 - \xi \sin^2 \theta' + 1$. This leads to

$$\frac{d\sigma}{dP} = \frac{r_e^2}{\epsilon^2 a_1} \frac{\Delta}{2\gamma^2 a \xi} \delta(\epsilon_1 - \tilde{\epsilon}_1). \quad (\text{A4})$$

In equation (2.13) we set $\beta_{\text{rel}} = 1$, $\mathcal{F}_{12} = a_1$, $n_1 = n_\gamma$, $n_2 = n_- + n_+$, $F_1 = F_\gamma$, $F_2 = F_e$, $\epsilon_2 = \gamma$, and $\delta_{12} = 0$. Clearly $d\Omega_1 d\Omega_2 = 2\pi d\mu d\mu' d\phi$, $a = 1 - \beta\mu'$, $a_1 = 1 - \beta\mu''$, where $\mu'' = \mu\mu' + [(1 - \mu^2)(1 - \mu'^2)]^{1/2} \cos \phi$, and $b = 1 - \mu$. These substitutions lead to equation (3.2).

A.2. BREMSSTRAHLUNG EMISSIVITY

Here we derive equation (3.9) and explain the notation used in that connection. We are interested in the processes in which two particles of momenta $p_i = \gamma_i(1, \beta_i)$, $i = 1, 2$ radiatively scatter on each other to produce a photon of momentum $p = \epsilon(1, \mathbf{k})$. Here $c\beta_i$ are the particle velocities in the C frame and γ_i are the corresponding Lorentz factors, ϵ is the energy of the emitted photon, and \mathbf{k} is its directional unit vector. Let μ , μ' , and μ'' be the cosines of the angles between the pairs of vectors (β_1, β_2) , (β_1, \mathbf{k}) , and (β_2, \mathbf{k}) , respectively. The angle between the planes formed by the pairs of vectors (β_1, β_2) and (β_1, \mathbf{k}) is defined to be ϕ . We have $\mu'' = \mu\mu' + [(1 - \mu^2)(1 - \mu'^2)]^{1/2} \cos \phi$. In equation (2.13), because of the isotropy of the distribution functions, we can write $d\Omega_1 d\Omega_2 = 2\pi d\mu d\Omega$, where $d\Omega$ is an infinitesimal solid angle around \mathbf{k} . We define $(d\sigma/d\epsilon)_i = \epsilon^2 \int d\Omega (d\sigma/dP)_i$. The case $i = 1$ refers to the $e^\pm - e^\pm$ process and the case $i = 2$ refers to the $e^\pm - e^\mp$ bremsstrahlung. It is shown by Haug (1975b) that

$$\left(\frac{d\sigma}{d\epsilon}\right)_i = \begin{cases} \frac{\alpha r_e^2 \epsilon}{\pi} \int d\Omega \frac{C_i}{\rho \Delta_i} & \text{if } \epsilon \leq \epsilon^*, \\ 0 & \text{otherwise,} \end{cases} \quad (\text{A5})$$

where $\Delta_1 = \omega(\omega^2 - 4)^{1/2}$, $\Delta_2 = 2(\zeta^2 - 1)^{1/2}$, and $\rho = [\omega^2 - 2(x_1 + x_2)]^{1/2}$, while $\omega = [2(\zeta + 1)]^{1/2}$, $\zeta = p_1 p_2 = \gamma_1 \gamma_2 (1 - \beta_1 \beta_2 \mu)$, $x_1 = p p_1 = \epsilon \gamma_1 (1 - \beta_1 \mu')$, and $x_2 = p p_2 = \epsilon \gamma_2 (1 - \beta_2 \mu'')$. Here α is the fine-structure constant. Finally,

$$\epsilon^* = \frac{\zeta - 1}{\gamma_1 + \gamma_2 - [(\gamma_1 + \gamma_2)^2 - 2(\zeta + 1)]^{1/2}}, \quad (\text{A6})$$

and

$$C_i = \left\{ \frac{\sqrt{\rho^2 - 4}}{\pi} \int A d\Omega' \right\}_i. \quad (\text{A7})$$

The cross section C_1 was computed by Haug (1975a, eq. [A1]) and C_2 by Haug (1985a, eq. [A1]). This latter cross section has some minor errors, and the corrections are given in Haug (1985b). For $C_{1,2}$ we have followed the notation of Haug except that $d\Omega'$ was called $d\Omega_{p_1}$ in C_1 and it was called $d\Omega_q$ in C_2 , in those papers. Going back to equation (2.13) we have to set $\delta_{12} = 1$ for the case $i = 1$. Hence $n_1 n_2 \rightarrow \frac{1}{2}(n_+^2 + n_-^2)$. In the second case $\delta_{12} = 0$ and $n_1 n_2 \rightarrow n_+ n_-$. We have $F_1 = F_2 = F_e$ and $\epsilon_i = \gamma_i$ for $i = 1, 2$. With these substitutions the desired result follows. In the present notation $\mathcal{F}_{12} = (\zeta^2 - 1)^{1/2} / \gamma_1 \gamma_2$ and $d\Omega = d\mu' d\phi$. The integration domain U is specified by $\gamma_{\min} \leq \gamma_{1,2} \leq \gamma_{\max}$, $-1 \leq \mu, \mu' \leq 1$, and $0 \leq \phi \leq 2\pi$, subject to the condition that $\epsilon^*(\gamma_1, \gamma_2, \mu) \geq \epsilon$.

A.3. COMPTON SCATTERING RATE FOR PAIRS

The cross section can be written as (see, e.g., JR80)

$$\sigma = \frac{r_e^2}{2\epsilon\gamma\rho_1} \int d\tau_f \delta^{(4)}(q + q_1 - p - p_1) X, \quad (\text{A8})$$

where $d\tau_f = d^3\mathbf{q} d^3\mathbf{q}_1$, $\mathbf{q} = \gamma\boldsymbol{\beta}$, $\mathbf{q}_1 = \epsilon\mathbf{k}$, while X is given by equation (4.3). Here $\rho_1 = p p_1 = q q_1$ and $\rho_2 = p q_1 = p_1 q$. In equation (A8) we can remove three of the delta functions by integrating over $d^3\mathbf{q}'$. Using the conservation of three-momentum we obtain $\epsilon^2 = (\gamma_1 \beta_1 + \epsilon_1 \mathbf{k}_1 - \gamma \boldsymbol{\beta})^2$ and $d\epsilon/d\gamma = (\gamma\beta - \gamma_1 \beta_1 \mu' - \epsilon_1 \mu)/\beta\epsilon$, while μ and μ' are defined in § 4. After some straight forward manipulations we find

$$\frac{d\sigma}{dP} = \frac{r_e^2}{2\gamma\epsilon\rho_1} \left| \frac{d\tilde{\epsilon}_1/d\gamma}{1 + d\epsilon/d\gamma} \right| X \delta(\epsilon_1 - \tilde{\epsilon}_1), \quad (\text{A9})$$

where $dP = d^3\mathbf{q} = \beta\gamma^2 d\gamma d\Omega$, and $d\Omega$ is the infinitesimal solid angle around the direction $\boldsymbol{\beta}$. Now in equation (2.13) we set $\beta_{\text{rel}} = 1$, $\mathcal{F}_{12} = a_1$, $n_1 = n_\gamma$, $n_2 = n_- + n_+$, $F_1 = F_\gamma$, $F_2 = F_e$, $\delta_{12} = 0$, and $\epsilon_2 = \gamma_1$. Clearly $a = 1 - \beta\mu$, $a_1 = 1 - \beta_1 \mu''$, where $\mu'' = \mu\mu' + [(1 - \mu^2)(1 - \mu'^2)]^{1/2} \cos \phi$, and $b = 1 - \beta\beta_1 \mu'$. Finally, $d\Omega_1 d\Omega_2 = 2\pi d\mu d\mu' d\phi$. These substitutions lead to equation (4.2).

A.4. PAIR CREATION RATE

Let $d\Omega_i$ be the infinitesimal solid angles around k_i for $i = 1, 2$. Infinitesimal solid angles around β and β_{cm} are denoted by $d\Omega$ and $d\Omega_{\text{cm}}$, respectively. In equation (2.13) we have $d\Omega_1 d\Omega_2 = 2\pi d\mu d\Omega$ because of the isotropy. We define $d\sigma/d\gamma = \beta\gamma^2 \int d\Omega(d\sigma/dP)$, where $dP = \beta\gamma^2 d\gamma d\Omega$. We set $n_1 = n_2 = n_\gamma$, $\delta_{12} = 1$, $\mathcal{F}_{12} = 1 - \mu$, and $F_1 = F_2 = F_\gamma$. It can be shown that

$$\frac{d\sigma}{d\gamma} = \int d\Omega_{\text{cm}} \frac{d\gamma_{\text{cm}}}{d\gamma} \left(\frac{d^2\sigma}{d\gamma d\Omega_{\text{cm}}} \right) = \int d\Omega_{\text{cm}} \frac{d\gamma_{\text{cm}}}{d\gamma} \left(\frac{d\sigma}{d\Omega_{\text{cm}}} \right) H(\epsilon_{\text{cm}}) \delta(\gamma_{\text{cm}} - \epsilon_{\text{cm}}), \quad (\text{A10})$$

where H is the Heaviside step function which is zero for negative arguments and is unity otherwise. The latter imposes the pair creation threshold. It can be easily seen that $d\Omega_{\text{cm}} = dz d\phi$. The delta function in the last equation ensures energy conservation. It can be written in the form $\delta(\gamma_{\text{cm}} - \epsilon_{\text{cm}}) = |dz/d\gamma_{\text{cm}}| \delta(z - \tilde{z})$, where $\tilde{z} = (\alpha_c \gamma_{\text{cm}} - \gamma) \Delta^{-1}$ and $\Delta = \beta_c \beta_{\text{cm}} \gamma_c \gamma_{\text{cm}}$. This is the solution to the equation $\gamma = \gamma_c \gamma_{\text{cm}} (1 - \beta_c \beta_{\text{cm}} z)$ (i.e., $p q = p_{\text{cm}} q_{\text{cm}}$). Finally, $|dz/d\gamma| = \Delta^{-1}$. After all these substitutions in equation (2.13) we arrive at equation (4.5).

A.5. BREMSSTRAHLUNG COOLING FUNCTIONS

Here we give the cooling functions used in equation (4.8). The energy radiated per unit time in e^\pm -proton collisions is given by

$$E_{ep}(\gamma) = cn_p \int_0^{\gamma^{-1}} d\epsilon \epsilon \left(\frac{d\sigma}{d\epsilon} \right)_{\text{proton}}, \quad (\text{A11})$$

where the protons are assumed to be at rest. Here ϵ is the energy of the emitted photon and $d\sigma/d\epsilon$ is the cross section (see, e.g., JR80). For E_{ee} we start from equation (3.5) of Haug (1975b). After some algebra we arrive at

$$E_{ee}(\gamma, \gamma') = \frac{c(n_+^2 + n_-^2)}{2n_e} \frac{\gamma + \gamma'}{\gamma\gamma'} \int_{-1}^1 d\mu p_c Q_{ee}(\epsilon_c, p_c), \quad (\text{A12})$$

where $\epsilon_c = [(\zeta + 1)/2]^{1/2}$, $p_c = [(\zeta - 1)/2]^{1/2}$, and $\zeta = \gamma\gamma'(1 - \beta\beta'\mu)$, while μ is the cosine of the interaction angle. Averaging over this angle (μ -integration) gave rise to the factor of half above. Presence of n_e in the denominator is a consequence of our definition of E_{ee} . The cooling function Q_{ee} , which is accurate to $\sim 6\%$ or better, is given by equation (3.15) of Haug (1975b). We reproduce it here for convenience:

$$Q_{ee} \approx 8\alpha r_e^2 \frac{p_c^2}{\epsilon_c} \left[1 - \frac{4}{3} \frac{P_c}{\epsilon_c} + \frac{2}{3} \left(2 + \frac{p_c^2}{\epsilon_c^2} \right) \ln(\epsilon_c + p_c) \right], \quad (\text{A13})$$

where α is the fine-structure constant. For $e^\pm e^\mp$ process we get

$$E_{ee}(\gamma, \gamma') = \frac{cn_+ n_-}{2n_e} \frac{\gamma + \gamma'}{\gamma\gamma'} \int_{-1}^1 d\mu p_c Q_{ee}(\epsilon_c, p_c). \quad (\text{A14})$$

The cooling function Q_{ee} is given by equations (26) and (28) of Haug (1985c). For the sake of convenience, we reproduce it here:

$$Q_{ee} = \begin{cases} \frac{32}{3} \alpha r_e^2 \sum_{i=0}^4 a_i p_c^i & \text{if } E_c \leq 300 \text{ KeV}, \\ 16\alpha r_e^2 [\epsilon_c \ln(\epsilon_c + p_c) - \frac{1}{6} \epsilon_c + \sum_{i=0}^2 b_i \epsilon_c^{-i}] & \text{otherwise}, \end{cases} \quad (\text{A15})$$

where $a_0 = 1.096$, $a_1 = -0.523$, $a_2 = 0.1436$, $a_3 = 1.365$, $a_4 = -0.532$, $b_0 = -0.726$, $b_1 = 1.575$, and $b_2 = -0.796$. Here $E_c = mc^2 \epsilon_c$.

A.6. LANDAU COLLISION INTEGRAL FOR COULOMB COLLISIONS

The flux vector (see Lifshitz & Pitaevskii 1981, hereafter LP81) is given by

$$S^i(\mathbf{p}) = \sum_{s=1}^2 \int d^3\mathbf{p}' \left[f_1(\mathbf{p}) \frac{\partial}{\partial p'^j} f_s(\mathbf{p}') - f_s(\mathbf{p}') \frac{\partial}{\partial p'^j} f_1(\mathbf{p}) \right] B^{ij}. \quad (\text{A16})$$

The superscripts i, j in this equation denote the components of three-vectors or tensors. In equation (A16) the summation over j is implicit. The components of momenta are given by $p^i = \gamma\beta^i$ and $p'^i = \gamma'\beta'^i$, for $i = 1, 2, 3$. We have $d^3\mathbf{p}' = \beta'\gamma'^2 d\Omega'$. Let $\zeta = \gamma\gamma'(1 - \beta\beta'\mu)$, where μ is the cosine of the interaction angle. The tensor B^{ij} (see LP81) is given by

$$B^{ij} = \frac{2\pi c r_e^2 \ln \Lambda_c \zeta^2}{\gamma\gamma'(\zeta^2 - 1)^{3/2}} [(\zeta^2 - 1)\delta^{ij} - \beta^i \beta^j \gamma^2 - \beta'^i \beta'^j \gamma'^2 + (\beta^i \beta'^j + \beta^j \beta'^i) \gamma\gamma' \zeta], \quad (\text{A17})$$

where we have made some slight modifications to take into account the dimensions of the distributions and the momenta. This tensor satisfies the identity $\sum_{j=1}^3 B^{ij}(\beta^j - \beta'^j) = 0$, for $i = 1, 2, 3$. For isotropic distributions we have $\partial f_s / \partial p^j = \beta^j \partial f_s / \partial \gamma$.

Using this fact and the previous identify we obtain

$$S_1^i = \int d\gamma' d\Omega' \beta' \gamma'^2 \mathcal{D}_1(\gamma, \gamma') \sum_{j=1}^3 B^{ij} \beta^j, \quad (\text{A18})$$

where

$$\mathcal{D}_1(\gamma, \gamma') = \sum_{s=1}^2 \left[f_1(\gamma) \frac{\partial}{\partial \gamma'} f_s(\gamma') - f_s(\gamma') \frac{\partial}{\partial \gamma} f_1(\gamma) \right]. \quad (\text{A19})$$

We choose a coordinate frame in which $\beta^1 = \beta$, $\beta^{2,3} = 0$, $\beta'^1 = \beta'\mu$, $\beta'^2 = \beta'(1 - \mu^2)^{1/2}$, and $\beta'^3 = 0$. Also $d\Omega' = 2\pi d\mu$. With these substitutions we find

$$\frac{\partial}{\partial t} f_1(\gamma) = -\beta \frac{\partial}{\partial \gamma} \int 2\pi d\mu d\gamma' \beta \beta' \gamma'^2 B \mathcal{D}_1(\gamma, \gamma'), \quad (\text{A20})$$

where $B = 2\pi c r_e^2 \ln \Lambda_C B_0$ and

$$B_0 = \frac{\zeta^2}{\gamma\gamma'(\zeta^2 - 1)^{3/2}} (\zeta^2 - 1 - \beta^2\gamma^2 - \beta'^2\gamma'^2\mu^2 + 2\beta\beta'\gamma\gamma'\mu\zeta). \quad (\text{A21})$$

The integral in equation (A20) is the Landau collision integral for small-angle deflections. This leads to the required result.

REFERENCES

- Baring, M. 1987a, MNRAS, 228, 681
 ———, 1987b, MNRAS, 228, 695
 Bisnovatyi-Kogan, G. S., Zeldovich, Y. B., & Sunyaev, R. A. 1971, AJ, 15, 17
 Björnsson, G., & Svensson, R. 1992, AJ, 394, 500
 Blumenthal, G. R., & Gould, R. J. 1970, Rev. Mod. Phys., 42, 237
 Carrigan, B. J., & Katz, J. I. 1992, AJ, 399, 100
 Chen, K., & Ruderman, M. 1993, AJ, 402, 264
 Coppi, P. S. 1992, MNRAS, 258, 657 (C92)
 Coppi, P. S., & Blandford, R. D. 1990, MNRAS, 245, 453 (CB90)
 de Groot, S. R., van Leeuwen, W. A., & van Weert, C. G. 1980, Relativistic Kinetic Theory (Amsterdam: North-Holland)
 Dermer, C. D. 1986, AJ, 307, 47
 Dermer, C. D., & Liang, E. P. 1989, AJ, 339, 512
 Fabian, A. C., Blandford, R. D., Guilbert, P. W., Phinney, E. S., & Cuellar, L. 1986, MNRAS, 221, 931
 Ghisellini, G. 1987, MNRAS, 224, 1
 Guilbert, P. W., & Stepney, S. 1985, MNRAS, 212, 523
 Haug, E. 1975a, Z. Naturforschung, 30a, 1099
 ———, 1975b, Z. Naturforschung, 30a, 1546
 ———, 1985a, Phys. Rev. D, 31, 2120
 ———, 1985b, Phys. Rev. D, 32, 1594(E)
 ———, 1985c, A&A, 148, 386
 ———, 1987, A&A, 178, 292
 ———, 1989, A&A, 218, 330
 Jauch, J. M., & Rohrlich, F. 1980, The Theory of Photons and Electrons (Berlin: Springer-Verlag) (JR80)
 Kusunose, M. 1987, AJ, 321, 186
 Landau, L. D., & Lifshitz, E. M. 1975, Classical Theory of Fields (New York: Pergamon)
 Lepage, P. 1978, J. Comp. Phys., 27, 192
 Lifshitz, E. M., & Pitaevskii, L. P. 1981, Physical Kinetics (New York: Pergamon) (LP81)
 Lightman, A. P. 1981, AJ, 244, 392
 Lightman, A. P. 1982, AJ, 253, 842
 Lightman, A. P., & Band, D. L. 1981, AJ, 251, 713
 Lightman, A. P., & Zdziarski, A. A. 1987, AJ, 319, 643
 Mészáros, P., & Rees, M. J. 1993a, AJ, 418, L59
 ———, 1993b, AJ, 405, 278
 Novikov, I. D., & Stern, B. E. 1986 in Structure and Evolution of Active Galactic Nuclei, ed. G. Giurcin, F. Mardirossian, M. Mezzetti, & M. Ramella (Dordrecht: Reidel), 149
 Padovani, P. 1996, preprint
 Piran, T., & Shaham, J. 1977, Phys. Rev. D, 16, 1615
 Podznyakov, L. A., Sobol, I. M., & Sunyaev, R. A. 1977, Soviet Astron., 21, 708
 Press, W. H., et al. 1992, Numerical Recipes (New York: Cambridge Univ. Press)
 Ramaty, R., McKinley, J. M., & Jones, F. C. 1982, AJ, 256, 238
 Ramaty, R., & Mészáros, P. 1981, AJ, 250, 384
 Rybicki, G. B., & Lightman, A. P. 1979, Radiative Processes in Astrophysics (New York: John Wiley), chap. 7
 Sikora, M. 1994, AJ Supp., 90, 923
 Stern, B. E. 1985, Soviet Astron., 29, 306
 Stern, B. E., Begelman, M. C., Sikora, M., & Svensson, R. 1995, MNRAS, 272, 291
 Sunyaev, R. A., et al. 1992 AJ, 389, L75
 Svensson, R. 1982a, AJ, 258, 321 (S82a)
 ———, 1982b, AJ, 258, 335 (S82b)
 ———, 1987, MNRAS, 227, 403
 ———, 1994, AJ Supp., 92, 585
 Tanaka, F., & Kusunose, M. 1985, Prog. Theor. Phys., 73, 1390
 Yahel, R. Z., & Brinkmann, W. 1981, AJ, 244, L7
 Zdziarski, A. A. 1980, Acta Astron., 30, 371
 ———, 1985, AJ, 289, 514
 ———, 1988, AJ, 335, 786
 ———, 1989, AJ, 342, 1108
 Zdziarski, A. A., Coppi, P. S., & Lamb, D. Q. 1990, AJ, 357, 149